Planar Isoperimetric Problems

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Overview

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Introduction: What is an Isoperimetric Problem?

- "**Isoperimetric**" means "**having the same perimeter**".
- The isoperimetric problem is to determine a plane figure of the largest possible area whose boundary has a specified length.
- Equivalently, among all closed curves in the plane enclosing a fixed area, which curve (if any) minimizes the perimeter?
- In fact, they are Dual Problems. Euler made this observation.

Earliest Days

- Isoperimetric Problem was first articulated by **Zenodorus**, a 2nd century B.C.E. Greek mathematician.
- Known through 4th century C.E. commenteries by **Theon of Alexandria** and **Pappus**.
- **Zenodorous**'s Contribution:
  - Among rectilinear figures with equal perimeter the one with most angles has the largest area.
  - The circle has greater area than any regular polygon of equal perimeter.
  - The equilateral and equiangular (i.e., regular) polygon has the greatest area of any polygon with the same perimeter and number of sides.

Dido's Problem

- **Dido** is the legendary founder and first queen of Carthage.
- She fled her native city, Tyre (Phoenicia, now Lebanon), to escape tyranny and tragedy, and arrived in North Africa and sought land from King Iarbas.
- Legend of the Oxhide: Asked for as much land as could be enclosed by an oxhide.
- Cleverly cut the oxhide into thin strips to encircle a large hill.
- Known as the Dido Problem in modern calculus of variations.
- The problem seeks to enclose the maximum area within a fixed boundary.

Illustration

Application of the Dido Maximum Principle

Medieval map of Cologne

Later On . . .

- Around 4th century C.E.: Pappus of Alexandria proposed among all circular segments with the same circumference, the semicircle has the largest area.
- 1691: Johann and Jakob Bernoulli argued over Johann's incorrect solution to the problem.
- 1744: Euler developed multiplier theory for solving the problem. Also, demonstrated that the circle is a necessary but not sufficient solution.
- 1838: Jacob Steiner first created a geometric proof for this problem, however it had a flaw...
- 1879: Weierstrass provided the first complete proof during his university lectures by showing the existence of a solution using analytical methods.

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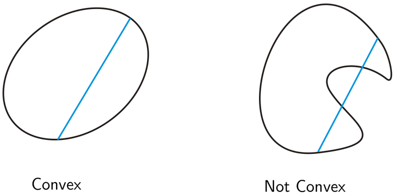
Preliminary Definitions

- A **Region** is a closure of a bounded open connected set whose boundary is a simple closed rectifiable curve.
- Notation: Length of the line segment AB is $l(AB)$ and angle ABC is $m(\angle ABC)$.
- **Convex Set:** For $x, y \in S$, $\alpha x + (1 - \alpha)x \in S$ for all $\alpha \in [0, 1]$.

Convex Set S



Preliminary Definitions

- **Jordan Curve Theorem:** A simple closed curve in the plane divides the plane into two regions, one compact and one noncompact, and is the common boundary of both regions.
- **Boundary of a Convex Set:** Let R be a convex set with boundary γ. Then γ is a simple, closed, continuous curve.
- Properties of Boundary of a Convex Set: At each of its points, it has a right and left hand tangent. The right (left) hand tangent at A is a ray from A tangent to γ in the right (left) hand direction.
- **Interior Angles:** The angles between these rays measured in the sector containing R. Since R is convex, the interior angle at every point will be at most 180°.

The Two Statements

Statement 1

Among all planar shapes with the same perimeter the circle has the largest area.

Statement 2

Among all planar shapes with the same area the circle has the shortest perimeter.

Duality of the Problem: The Equivalence between the two Statements

The shape that minimises perimeter given a fixed area also yields the maximum area among all shapes with its perimeter.

(1) $\implies$ (2):

- Assume Statement 1 holds and Statement 2 is False.
- Say for a given circle C, $\exists$ figure F such that: Perimeter(F) $<$ Perimeter(C), where Area(C) $=$ Area(F).
- Shrink C to a smaller circle C' such that Perimeter(C') $=$ Perimeter(F).
- Area(C') $<$ Area(C) $=$ Area(F).
- Contradiction. (By statement 1, Area(C') should be larger.)

Duality of the Problem: The Equivalence between the two Statements

(2) $\implies$ (1):

- Assume Statement 2 holds and Statement 1 is False.
- Say for a given circle C, $\exists$ figure G such that: Area(G) $>$ Area(C) where Perimeter(G) $=$ Perimeter(C).
- Expand C to a bigger circle C' such that Area(C') $=$ Area(G).
- Perimeter(C') $>$ Perimeter(C) $=$ Perimeter(G).
- Contradiction. (By statement 2, Perimeter(C') should be smaller.)

Proof of Statement 1: An Outline

- For the "proof" we assume, given a fixed perimeter, such a planar shape with "a maximum area" exists.
- Firstly, observe that the figure that solves statement 1, must be convex.
- Why? Steiner's Symmetrization Argument.
- It comes with three main steps, discussed next!

Steiner's Symmetrization: Step 1

Steiner was one of the key founders of modern synthetic geometry. His symmetrization proof consists of the following steps:

- The curve must be convex. Proof:
  - If not, $\exists$ two points from the figure such that the connecting segment would not itself lie inside our shape (fig 1).
  - Reflect the region between the segment & the boundary (fig 2).
  - Continue this process till we make the figure convex. The resulting figure has the same perimeter but with more area.

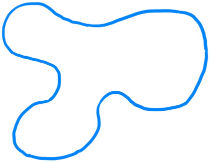


Fig 1

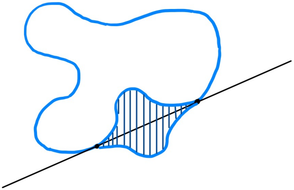


Fig 2

Steiner's Symmetrization: Step 2

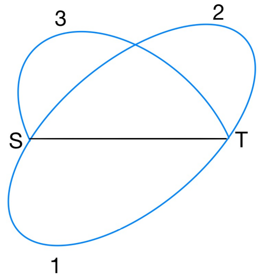
- Perimeter bisector divides the curve into equal areas and we can therefore symmetrize across the bisector.

Lemma 1

Let two points S and T be selected on the boundary of the shape solving Statement 1 such that the points divide the perimeter into two equal parts. Then the segment ST must divide the area of the shape into two equal parts.

Proof of Lemma 1

- Say, the shape S1T2 is the solution to Statement 1.
- Let's say WLOG, Area(S1T) $>$ Area(S2T).
- Reflect S1T to get the shape S1T3.
- Now we get, Area(S1T3) $>$ Area(S1T2), however Perimeter(S1T2) $=$ Perimeter(S1T3).
- Contradiction to statement 1.



Steiner's Symmetrization: Step 3

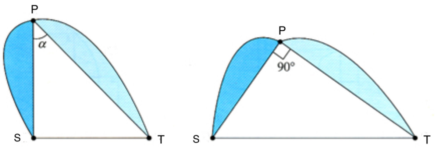
- All inscribed angles determined by the perimeter bisector must be right angles.

Lemma 2

Consider all the arcs with a given length and the endpoints S and T on a fixed straight line. The curve that encloses the maximum area between the curve and the straight line is a semicircle.

Proof of Lemma 2

- Suffices to show that every angle inscribed into the arc is 90°.
- If for a point P on the arc $\widehat{ST}$, $\angle SPT \neq 90^\circ$, slide either point S or point T along the straight line $\overline{ST}$ until the angle becomes 90°.
- Among all triangles with two given sides, the right angled one has the largest area. (Area of $\Delta = \frac{1}{2}$.base.height)
- Since the area of the two blue regions did not change but the area of the triangle grew, the area between $\widehat{SPT}$ and the line $\overline{ST}$ has increased.
- Conclusion: Optimal figure with the maximum area is a semi-circle.



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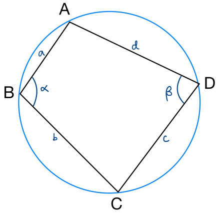
Quadrilaterals $n = 4$

Theorem 11

A quadrilateral with given sides has the greatest area when it can be inscribed in a circle.

Proof:

- Consider a quadrilateral with sides a, b, c, d, perimeter L and area T.
- $\angle ABC = \alpha$, $\angle ADC = \beta$.
- Say, $l(BD) = f$.
- Area(ABCD)= Area(ABC) + Area(ACD).
- $2T = ab\sin(\alpha) + cd\sin(\beta)$.
- This gives: $16T^2 = 4a^2b^2 + 8abcd\sin\alpha\sin\beta + 4c^2d^2\sin^2\beta$.



Quadrilaterals $n = 4$

- Law of cosine gives: $a^2 + b^2 - 2ab\cos\alpha = f^2 = c^2 + d^2 - 2cd\cos\beta$.
- $a^2 + b^2 - c^2 - d^2 = 2ab\cos\alpha - 2cd\cos\beta$.
- Take square and add to $16T^2$.
- $16T^2 + (a^2+b^2-c^2-d^2)^2 = 4a^2b^2 + 4c^2d^2 - 8abcd\cos(\alpha+\beta)$.
- T is largest when $\alpha + \beta = \pi$.
- ABCD is a cyclic quadrilateral.

General n-sided Polygon

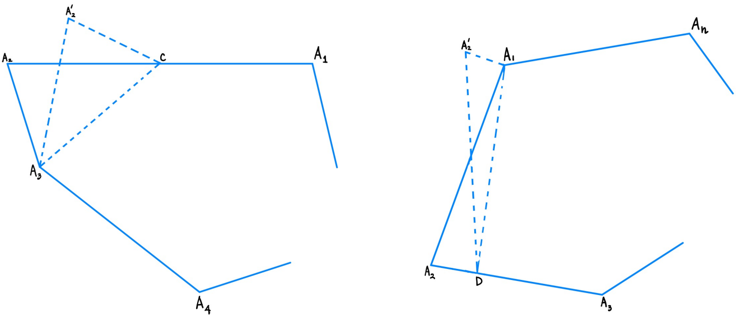
Problem 9

Among all n-sided polygons of perimeter p, the regular n-gon has the maximum area.

Proof:

- Step 1: Solution P_n must be equilateral.
  - Let $P_n = A_1A_2 \dots A_n$ is not equilateral. Say, $l(A_1A_2) > l(A_2A_3)$.
  - Choose C on A_1A_2 such that $l(A_2A_3) < l(A_2C) < l(A_2A_1)$.
  - $m(\angle A_2A_3C) > m(\angle A_2CA_3)$.
  - Reflect ΔA_2A_3C in the perpendicular bisector of A_3C. And A_2' is reflection of A_2.
  - $m(\angle A_2'CA_3) + m(\angle A_1CA_3) > 180^\circ$.
  - $(n+1)$-sided polygon $P' = A_1CA_2'A_3 \dots A_n$ is not convex.
  - Perimeter(P_n) = Perimeter(P'). Draw A_1A_2'.
  - n-sided polygon $P'' = A_1A_2'A_3 \dots A_n$ has perimeter smaller than P_n and area greater than P_n. A contradiction.

General n-sided Polygon



General n-sided Polygon

- Step 2: Solution P_n must be equiangular.
  - Say, $m(\angle A_n A_1 A_2) > m(\angle A_1 A_2 A_3)$.
  - $\exists\ 0 < \alpha < \frac{1}{2}[m(\angle A_n A_1 A_2) - m(\angle A_1 A_2 A_3)]$.
  - Choose D on $A_2 A_3$ such that $m(\angle A_2 A_1 D) = \alpha$.
  - Can show : $m(\angle A_3 D A_1) < m(\angle A_n A_1 D)$.
  - Reflect $\Delta A_1 A_2 D$ in the perpendicular bisector of $A_1 D$. A_2' be the image of A_2.
  - Since $m(\angle A_2' A_1 D) = m(\angle A_2 D A_1)$, so :

$$\begin{aligned} m(\angle A_n A_1 D) + m(\angle A_2' A_1 D) &= m(\angle A_n A_1 D) + m(\angle A_2 D A_1) \\ &> m(\angle A_3 D A_1) + m(\angle A_2 D A_1) \\ &= 180^\circ. \end{aligned}$$

  - The polygon $P' = A_1 A_2' D A_3 \ldots A_n$ is not convex.
  - Again draw $A_n A_2'$. $P'' = A_2' D A_3 \ldots A_n$ has n sides with area greater than and perimeter smaller than P_n. A contradiction.

n-sided Polygon inscribed in a Circle: Few more results

A few more results mentioned without proof:

- Among n-sided polygons inscribed in a circle, the regular n-gon has the largest area.
- Among n-sided polygons of given side lengths, the one inscribed in a circle has the largest area.

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The Insufficiency of Steiner's Proof

- But are we entitled to conclude that a maximum area region actually exists?
- Enter **Karl Weierstrass.**
- He provides an analytic solution to the problem by defining area and length in terms of integrals and by using **calculus of variations**.
- It provides an equation of the form of a circle and the length of the radius which optimizes the area given the length of the perimeter.

Isoperimetric Inequality for the Plane

Among all regions in the plane, enclosed by a piecewise C^1 boundary curve, with area A and perimeter L,

$$L^2 \geq 4\pi A$$

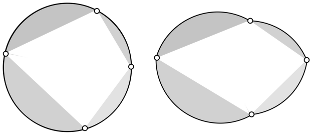
If equality holds, then the region is a circle.

Proof Using Theorem 11

- Assume that the optimal convex shape Ω having the maximum area for a fixed perimeter isn't a circle.
- Find four points on the boundary A,B,C,D such that quadrilateral *ABCD* is not cyclic.
- Brahmagupta's formula for general quadrilateral:

$$T = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta}$$

  where θ is half the sum of two opposite angles.
- Area maximizes when quadrilateral is cyclic.
- Deform portions of Ω lying outside of ABCD such that area increases while keeping perimeter fixed. Contradiction!
- Pick a figure with area A, perimeter L which isn't a circle.
- $A \leq \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$.



A Sufficient Analytic Proof of the Inequality

- Let C be a smooth regular simple closed curve in the plane, with perimeter L and area A.
- For each $\epsilon > 0$, inscribe in C a polygon P_ϵ with perimeter L_ϵ satisfying $|L_\epsilon - L| < \epsilon$ with area A_ϵ satisfying $|A_\epsilon - A| < \epsilon$.
- For polygon P_ϵ we already have, $L_\epsilon^2 \geq 4\pi A_\epsilon$.
- Why? Proof:
  - Say for n-gon with side length s and apothem a, $L = n \cdot s$ and $A = \frac{n \cdot s \cdot a}{2}$, where $a = \dfrac{s}{2\tan(\frac{\pi}{n})}$.
  - Inequality easily follows using the fact that: $\tan(x) \geq x$ for $0 \leq x < \frac{\pi}{2}$, here for $n \geq 2$, use $x = \frac{\pi}{n} < \frac{\pi}{2}$.
- Let $\epsilon \to 0$ to get the inequality for a circular region.
- So the maximizer exists.

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Physical Manifestations of the Isoperimetric Inequality

- Water drops assume a symmetric round shape (sphere) due to surface tension minimizing the surface area for a fixed volume.
- Cats curl up to minimize exposed surface area and conserve heat.
- Honeybees build hexagonal cells in hives as this shape is efficient and minimizes materials while maximizing space.
- Principle of Least Action in Physics: Minimizing effort while maximizing effect.

THANK YOU