THE DIMENSION OF POSETS WITH PLANAR COVER GRAPHS

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Outline

Main Theorem

If P is a poset with a planar comparability graph G_P, then $\dim(P) \leq 4$.

Definitions

Vertex-Edge-Face Poset Q_G

Let $G = (V, E, F)$ be a planar graph. The vertex-edge-face poset associated with the drawing $\mathcal{D}$ of G is a height 3 poset Q_G where

- minimal elements are vertices of G
- maximal elements are faces of G
- $v \in V$ and $e \in E$ then $v < e$ iff v is one endpoint of e
- $e \in E$ and $f \in F$ then $e < f$ iff e is on the boundary of f

Definitions

"Split" of a subset (Kimble 1973)

$S \neq \emptyset$ is a subset of the ground set of a poset P. The split of S is the poset $Q = \text{split}(S, P)$ whose ground set is:

$$(P - S) \cup \{x' : x \in S\} \cup \{x'' : x \in S\}.$$

The order relation on Q is defined as follows:

- $Q|_{P-S} = P|_{P-S}$
- $\{x' : x \in S\}$ are minimal in Q
- $\{x'' : x \in S\}$ are maximal in Q
- If $x, y \in S$, then $x' < y''$ in Q when $x \leq y$ in P
- If $x \in S$; $u \in P - S$, then $x' < u$ in Q when $x < u$ in P
- If $x \in S$; $u \in P - S$, then $u < x''$ in Q when $u < x$ in P

Preliminary Results

Theorem 1 (Dimension bound on a Split)

Let S be a non-empty subset of the ground set of a poset P and let Q be the split of S. Then $\dim(P) \leq \dim(Q) \leq 1 + \dim(P)$.

Theorem 2 (Brightwell & Trotter 1993)

Let $\mathcal{D}$ be a planar drawing of graph G and let Q_G be the vertex-edge-face incidence poset Q_G determined by $\mathcal{D}$. Then,

$$\dim(Q_G) \leq 4.$$

Proof of Main Theorem

1. This is a proof by contradiction.
2. Assumptions on P :
    - P has a planar comparability graph G
    - $\dim(P) \geq 5$
    - Partition of ground set : $P = X \cup S \cup Y$
    - minimal element set : X
    - maximal element set : Y
    - P is a minimal choice for the quantity $q(P) = |X| + |Y| + 3|S|$
    - Clearly, P is connected

Proof of Main Theorem

Fix a planar drawing $\mathcal{D}$ with no edge crossings of the comparability graph G_P. Now we analyze S.

1. **Case 1: $S = \emptyset$.**
   - height of P is 2
   - G_P is essentially a planar cover graph

Theorem 3

If P is a poset with a planar cover graph and the height of P is at most 2, then

$$\dim(P) \leq 4.$$

So we are not in this case.

Proof of Main Theorem

2. **Case 2: $S \neq \emptyset$.** Define for a fixed $s \in S$:
   - $D(s) = \{u \in P : u < s\}$
   - $U(s) = \{v \in P : s < v\}$
   - $D(s), U(s) \neq \emptyset$.
   - height of P is 3.

Proof of Main Theorem

a. **Subcase 1:**
$\min\{|D(s)|, |U(s)|\} \geq 2$; $\max\{|D(s)|, |U(s)|\} \geq 3$.

   - Comparability graph G_P contains $K_{3,3}$ (non-planar).

A Contradiction.

Proof of Main Theorem

b. **Subcase 2: $\min\{|D(s)|, |U(s)|\} = 1$; $|D(s)| = 1$.**

- Say, $\{x\} = D(s)$.
- If Q is split of $s \in S \implies \dim(Q) \geq \dim(P) \geq 5$.
- $q(Q) < q(P)$. (why?)
- Q has a planar comparability graph. (G_Q can be constructed from G_P)

A Contradiction on the choice for P with $q(P)$ minimal.

Proof of Main Theorem

- Thus $\min\{|D(s)|, |U(s)|\} \geq 2$ and $\max\{|D(s)|, |U(s)|\} \leq 2$
- Thus $|D(s)| = 2$
- Dual Argument shows: $|U(s)| = 2$

Proof of Main Theorem

- **Subcase 3: $|D(s)|, |U(s)| = 2$.**
  - $D(s) = \{x_1, x_2\} \subseteq X$, $U(s) = \{y_1, y_2\} \subseteq Y$.
  - Two planar drawings for $\mathcal{D}$ of P.
    - ★ Block Position.
    - ★ Alternate Position.
  - Block Position is non-planar.
  - Alternate Position is planar.

Proof Strategy

We have: A fixed planar drawing $\mathcal{D}$ of G_P of height 3 in alternate position. $P = X \cup S \cup Y$.

Claim: P is isomorphic to a subposet of some Q'_G poset for a planar multigraph G'.

Proof Skeleton:

- Starting with $\mathcal{D}$ create a planar drawing $\mathcal{D}'$ without edge crossings which corresponds to a Q'_G.
- Argue using Trotter 1993 Theorem that $\dim(Q'_G) \leq 4$.
- Thus $\dim(P)$ must be at most 4.

(ϵ, δ) Construction of $\mathcal{D}'$

- vertex set of $\mathcal{D}'$ is X
- every $s \in S$ corresponds to edge e_s in $\mathcal{D}'$
- there can be other edges in $\mathcal{D}'$
- every $y \in Y$ associates to a face F_y in $\mathcal{D}'$
- If $x \in X$, $s \in S$ then $x < s$ in P iff x is endpoint of e_s
- If $x \in X$, $y \in Y$ then $x < y$ in P iff x is on the boundary of F_y
- If $s \in S$, $y \in Y$ then $s < y$ in P iff e_s is on the boundary of F_y
- loops and multiedges allowed in $\mathcal{D}'$ of G'
- ϵ is the min distance in $\mathcal{D}$ between distinct vertices of G_P
- δ is a small angle chosen to avoid edge crossings in $\mathcal{D}'$

Two cases of the Construction

1. **Case 1:** $y \in Y$ has a $s \in S$ such that $D(s) = \{x_1, x_2\}$, $x_i < s < y$ for $i = 1, 2$
2. **Case 2:** $y \in P$, $\exists$ x such that y covers x in P

References

1. Felsner, S., Trotter, W.T. & Wiechert, V. The Dimension of Posets with Planar Cover Graphs. Graphs and Combinatorics 31, 927–939 (2015). https://doi.org/10.1007/s00373-014-1430-4

THANK YOU