An Introduction to Manifolds

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ABSTRACT

This project is about the extension of calculus from curves and surfaces to higher dimensions. The higher-dimensional analogues of smooth curves and surfaces are called manifolds. Topological invariant of a manifold is a property that remains unchanged under a homeomorphism.

In the second chapter on Euclidean Spaces, we recast calculus on $\mathbb{R}^n$ in a coordinate-free way suitable for generalization to manifolds. We do this by giving meaning to the symbols dx, dy, and dz, so that they assume a life of their own, as differential forms. We begin with the notion of directional derivative on $\mathbb{R}^n$ and generalize this notion by introducing equivalence relation on C^∞ functions in the neighborhood of a point p and call the linear map of directional derivative as derivation at p. Next we discuss multilinear functions.

In the next chapter we rigorously define the notion of a topological manifold. Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. It is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset of $\mathbb{R}^n$. The coordinates on a chart allow one to carry out computations as though in a Euclidean space, so that many concepts from $\mathbb{R}^n$, such as differentiability, point-derivations, tangent spaces, and differential forms, carry over to a manifold.

Next we define the notion of a smooth manifold and smooth maps between two manifolds. Using coordinate charts, one can transfer the notion of smooth maps from Euclidean spaces to manifolds. By the C^∞ compatibility of charts in an atlas, the smoothness of a map turns out to be independent of the choice of charts and is therefore well defined. We give various criteria for the smoothness of a map.

Next we discuss the inverse function theorem which provides an answer to the following question : Given n smooth functions $F^1, \ldots, F^n$ in a neighborhood of a point p in a manifold N of dimension n, one would like to know whether they form a coordinate system, possibly on a smaller neighborhood of p. This is equivalent to whether $F = (F^1, \ldots, F^n) : N \to \mathbb{R}^n$ is a local diffeomorphism at p.

In next chapter we discuss the notion of tangent space to a manifold. By definition, the tangent space to a manifold at a point is the vector space of derivations at the point. A smooth map of manifolds induces a linear map, called its differential, of tangent spaces at corresponding points. In local coordinates, the differential is represented by the Jacobian matrix of partial derivatives of the map. In this sense, the differential of a map between manifolds is a generalization of the derivative of a map between Euclidean spaces.

The next chapter discusses immersion, submersion, embedding maps on a manifold and introduces the concept of a submanifold. Every manifold can be realized as a submanifold of a Euclidean space. One of the ways of showing that a given topological space is a manifold is by checking directly that the space is Hausdorff, second countable, and has a C^∞ atlas. As a topological space, a regular submanifold of N is required to have the subspace topology.

In the final chapter we discuss Differential forms as generalizations of real-valued functions on a manifold. In calculus, one limits oneself to curves and surfaces in $\mathbb{R}^3$. But in manifold theory we deal more generally with k manifolds in $\mathbb{R}^n$. In this project we limit our discussion only to introducing differential 1 form on manifolds.

Our final topic is Partition of Unity. A partition of unity on a manifold is a collection of non negative functions that sum to 1. It serves as a bridge between global and local analysis on a manifold. This is useful because while there are always local co-ordinates on a manifold, there may be no global coordinates.

DECLARATION BY THE CANDIDATE

I, Sayantani Bhattacharya, do hereby declare that the subject matter in this project report, is the record of reading work, done by me during the 4th semester of my M. Sc. course with full integrity, honesty and concentration under the immaculate supervision of Dr. Deepjyoti Goswami, Department of Mathematical Sciences and submitted to the Department of Mathematical Sciences, Tezpur University in fulfilment of the requirements for completing the course **MS 515: Project**.

Date:

Place: Tezpur

Signature of the Candidate

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List of Symbols

$\mathbb{R}^n$	Real Coordinate Space of dimension n
C^∞	Class of smooth or infinitely differentiable functions
$\dfrac{\partial f}{\partial x^i}$	partial derivative with respect to x^i
$f^{(k)}(x)$	the k th derivative of $f(x)$
$T_p(\mathbb{R}^n)$	Tangent space at p of in $\mathbb{R}^n$
$D_v f$	directional derivative of f in the direction of v at p
C_p^∞ or $C_p^\infty(\mathbb{R}^n)$	algebra of germs of C^∞ functions at p in $\mathbb{R}^n$
$\mathcal{D}_p(\mathbb{R}^n)$	vector space of derivations at p in $\mathbb{R}^n$
δ^i_j	Kronecker Delta
Hom(V,W)	vector space of linear maps $f : V \to W$
$V^\vee = \mathrm{Hom}(V, \mathbb{R})$	dual of a vector space
$\Omega^k(U)$	the vector space of C^∞ k-forms on U
σf	a function f acted on by a permutation σ
$f \otimes g$	tensor product of multilinear functions f and g
$f \wedge g$	wedge product of multicovectors f and g
e_I	k tuple $(e_{i_1}, \ldots, e_{i_k})$
α^I	k-covectors $\alpha^{i_1} \wedge \ldots \alpha^{i_k}$
$T_p^*(\mathbb{R}^n)$	cotangent space to $\mathbb{R}^n$
df	differential of a function
$\omega(X)$	the function $p \mapsto \omega(p)$
$d\omega$	exterior derivative of omega
$\{U_\alpha\}_{\alpha \in A}$	open cover
(U, ϕ)	chart or coordinate open set
S_k	group of all permutation of the set $\{1, \ldots, k\}$
$\mathbf{1}_U$	identity map on U
$A_k(V)$	vector space of alternating k-linear functions on V
$L_k(V)$	vector space of all k tensors
$J(f) = [\partial F^i / \partial x^j]$	Jacobian Matrix
$C_p^\infty(M)$	germs of C^∞ functions at p in M
$T_p(M)$	tangent space to M at p
$\partial/\partial x^i\big\rvert_p$	coordinate tangent vector at p
$d/dt\big\rvert_p$	coordinate tangent vector of a 1-dimensional manifold
$F_{*,p}$	differential of F at p
$c(t)$	curve in a manifold
$c'(t) = c\left(\dfrac{d}{dt}\bigg\rvert_{t_0}\right)$	velocity vector of a curve
$\dot{c}(t)$	derivative of a real-valued function
$f^{-1}(\{c\})$	level set

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Chapter 1

Introduction

The higher-dimensional analogues of smooth curves and surfaces are called manifolds. The constructions and theorems of vector calculus become simpler in the more general setting of manifolds; gradient, curl, and divergence are all special cases of the exterior derivative, and the fundamental theorem for line integrals, Green's theorem, Stokes's theorem, and the divergence theorem are different manifestations of a single general Stokes's theorem for manifolds. Higher-dimensional manifolds arise even if one is interested only in the three-dimensional space that we inhabit. For example, if we call a rotation followed by a translation an affine motion, then the set of all affine motions in $\mathbb{R}^3$ is a six- dimensional manifold. Moreover, this six-dimensional manifold is not $\mathbb{R}^6$.

Like most fundamental mathematical concepts, the idea of a manifold did not originate with a single person, but is rather the distillation of years of collective activity. In his masterpiece Disquisitiones generales circa superficies curvas ("General Investigations of Curved Surfaces") published in 1827, Carl Friedrich Gauss freely used local coordinates on a surface, and so he already had the idea of charts. Moreover, he appeared to be the first to consider a surface as an abstract space existing in its own right, independent of a particular embedding in a Euclidean space. Bernhard Riemann's inaugural lecture Über die Hypothesen, welche der Geometrie zu Grunde liegen ("On the hypotheses that under lie geometry") in Göttingen in 1854 laid the foundations of higher-dimensional differential geometry. Indeed, the word "manifold" is a direct translation of the German word "Mannigfaltigkeit," which Riemann used to describe the objects of his inquiry. This was followed by the work of Henri Poincaré in the late nineteenth century on homology, in which locally Euclidean spaces figured prominently. The late nineteenth and early twentieth centuries were also a period of feverish development in point-set topology. It was not until 1931 that one finds the modern definition of a manifold based on point-set topology and a group of transition functions.

The Euclidean space $\mathbb{R}^n$ is the simplest prototype of all manifolds. Not only is it the simplest, but locally every manifold looks like $\mathbb{R}^n$. Aside from Euclidean spaces themselves, are smooth plane curves such as circles and parabolas, and smooth surfaces $\mathbb{R}^3$ such as spheres, tori, paraboloids, ellipsoids, and hyperboloids. Higher-dimensional examples include the set of unit vectors in $\mathbb{R}^{n+1}$ (the n-sphere) and graphs of smooth maps between Euclidean spaces.

Many important applications of manifolds involve calculus. For example, the application of manifold theory to geometry involves the study of such properties as volume and curvature. Typically, volumes are computed by integration, and curvatures are computed by formulas involving second derivatives, so to extend these ideas to manifolds would require some means of making sense of differentiation and integration on a manifold. The application of manifold theory to classical mechanics involves solving systems of ordinary differential equations on manifolds, and the application to general relativity (the theory of gravitation) involves solving a system of partial differential equations.

Because integration of functions on a Euclidean space depends on a choice of coordinates and is not invariant under a change of coordinates, it is not possible to integrate functions on a manifold. The highest possible degree of a differential form is the dimension of the manifold. Among differential forms, those of top degree turn out to transform correctly under a change of coordinates and are precisely the objects that can be integrated. The theory of integration on a manifold would not be possible without differential forms. Very loosely speaking, differential forms are whatever appears under an integral sign. In this sense, differential forms are as old as calculus, and many theorems in calculus such as Cauchy's integral theorem or Green's theorem can be interpreted as statements about differential forms. Although it is difficult to say who first gave differential forms an independent meaning, Henri Poincaré and Élie Cartan are generally both regarded as pioneers in this regard. In the paper published in 1899, Cartan defined formally the algebra of differential forms on $\mathbb{R}^n$ as the anticommutative graded algebra over $\mathbb{C}^\infty$ functions generated by $dx_1, \ldots, dx_n$ in degree 1. In the same paper one finds for the first time the exterior derivative on differential forms.

Chapter 2

Preliminary Results on Euclidean Spaces

Recall that a *secant plane* to a surface in $\mathbb{R}^3$ is a plane determined by three points of the surface. As the three points approach a point p on the surface, if the corresponding secant planes approach a limiting position, then the plane that is the limiting position of the secant planes is called the **tangent plane** to the surface at p. Intuitively, the tangent plane to a surface at p is the plane in $\mathbb{R}^3$ that just "touches" the surface at p. A vector at p is tangent to a surface in $\mathbb{R}^3$ if it lies in the tangent plane at p.

Our goal in this section is to find a characterization of tangent vectors in $\mathbb{R}^n$ that will generalize to manifolds.

2.1 Directional Derivative

In calculus we visualize the tangent space at p in $\mathbb{R}^n$ denoted by $T_p(\mathbb{R}^n)$ as the vector space of all arrows emanating from p. The line through a point $p = (p^1, \ldots, p^n)$ with direction $v = < v^1, \ldots, v^n >$ in $\mathbb{R}^n$ has parametrization :

$$c(t) = (p^1 + tv^1, \ldots, p^n + tv^n).$$

If f is C^∞ in a neighborhood of p in $\mathbb{R}^n$ and v is a tangent vector at p, the **directional derivative** of f in the direction v at p is defined to be

$$D_v f \;=\; \lim_{t\to 0} \frac{f(c(t)) - f(p)}{t} \;=\; \left.\frac{d}{dt}\right|_{t=0} f(c(t))$$

By the chain rule,

$$D_v f \;=\; \sum_{i=1}^{n} \frac{dc^i}{dt}(0)\frac{\partial f}{\partial x^i}(p) \;=\; \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p)$$

The association $v \mapsto D_v$ of the directional derivative D_v to a tangent vector v offers a way to characterize tangent vectors as certain operators on functions.

2.1.1 Derivation at a Point

As long as two functions agree on some neighborhood of a point p, they will have the same directional derivatives at p. This suggests that we introduce an equivalence relation on the C^∞ functions defined in some neighborhood of p. Consider the set of all pairs (f, U), where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. We say that (f, U) is equivalent to (g, V) if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W . This is clearly an equivalence relation because it is reflexive, symmetric, and transitive. The equivalence class of (f, U) is called the **Germ** of f at p. We write $C_p^\infty(\mathbb{R}^n)$, or simply C_p^∞ for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

The addition and multiplication of functions induce corresponding operations on C_p^∞ , making it into an algebra over $\mathbb{R}$.

For each tangent vector v at a point p in $\mathbb{R}^n$, the directional derivative at p gives a map of real vector spaces

$$D_v : C_p^\infty \to \mathbb{R}$$

D_v is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$D_v(fg) \;=\; (D_v f)g(p) + f(p)D_v g$$

Any linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule is called a **derivation** at p or a point-derivation of C_p^∞. Recall $T_p(\mathbb{R}^n)$ is the tangent space at p of $\mathbb{R}^n$. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This set is in fact a real vector space, since the sum of two derivations at p and a scalar multiple of a derivation at p are again derivations at p. Thus far, we know that directional derivatives at p are all derivations at p, so there is a map,

$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n),$$

$$v \mapsto D_v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p$$

Lemma 2.1.1. If D is a point-derivation of C_p^∞ , then $D(c) = 0$ for any constant function c.

Proof. By R-linearity, $D(c) = cD(1)$. So it suffices to prove that $D(1) = 0$. By the Leibniz rule,

$$D(1) = D(1 \times 1) = D(1) \times 1 + 1 \times D(1) = 2D(1).$$

Subtracting $D(1)$ from both sides gives $0 = D(1)$.

Recall that, an *isomorphism* is a mapping between two structures of the same type that can be reversed by an inverse mapping. Informally, an isomorphism is a map that preserves sets and relations among elements.

Theorem 2.1.2. The linear map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ is an isomorphism of vector spaces.

Proof. To prove injectivity, suppose $D_v = 0$ for $v \in T_p(\mathbb{R}^n)$. Applying D_v to the coordinate function x^j gives

$$0 = D_v(x^j) = \sum_i v^i \frac{\partial}{\partial x^i}\bigg|_p x^j = \sum_i v^i \delta_i^j = v^j$$

Hence, $v = 0$ and ϕ is injective.

To prove surjectivity, let D be a derivation at p and let (f, V) be a representative of a germ in C_p^∞ (the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p). Making V smaller if necessary, we may assume that V is an open ball, hence star-shaped (a set in $\mathbb{R}^n$ is called star shaped if there exists an x_0 in S such that for all x in S the line segment from x_0 to x is in S). By Taylor's theorem with remainder there are C^∞ functions $g_i(x)$ in a neighborhood of p such that

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)\ ,\ g_i(p) = \frac{\partial f}{\partial x^i}(p)$$

Applying D to both sides and noting that $D(f(p)) = 0$ and $D(p^i) = 0$, we get by the Leibniz rule :

$$Df(x) = \sum (Dx^i) g^i(p) + \sum (p^i - p^i) Dg^i(x) = \sum (Dx^i) \frac{\partial f}{\partial x^i}(p)$$

This proves that $D = D_v$ for $v = < Dx^1, ..., Dx^n >$.

Under the vector space isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $\{e^1, \ldots, e^n\}$ for $T_p(\mathbb{R}^n)$ corresponds to the set $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ of partial derivatives.

2.2 Vector Field

A vector field X on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector X_p in $T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\{\partial/\partial x^i|_p\}$, the vector X_p is a linear combination

$$X_p = \sum a^i(p) \frac{\partial}{\partial x^i}\bigg|_p , \quad p \in U, \quad a^i(p) \in \mathbb{R}.$$

We say that the vector field X is C^∞ on U if the coefficient functions a^i are all C^∞ on U.

One can identify vector fields on U with column vectors of C^∞ functions on U:

$$X = \sum a^i \frac{\partial}{\partial x^i} \leftrightarrow \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}$$

The ring of C^∞ functions on an open set U is commonly denoted by $C^\infty(U)$ or $\mathcal{F}(U)$. Multiplication of vector fields by functions on U is defined pointwise :

$$(fX)_p = f(p)X_p, \quad p \in U.$$

Clearly, if $X = \sum a^i \partial/\partial x^i$ is a C^∞ vector field and f is a C^∞ function on U, then $fX = \sum (fa^i) \partial/\partial x^i$ is a C^∞ vector field on U. Thus, the set of all C^∞ vector fields on U, denoted by $X(U)$, is a vector space over $\mathbb{R}$.

2.2.1 Vector Fields as Derivatives

If X is a C^∞ vector field on an open subset U of $\mathbb{R}^n$ and f is a C^∞ function on U, we define a new function Xf on U by

$$(Xf)(p) = X_p f \qquad \text{for any} \quad p \in U.$$

Writing $X = \sum a^i \partial/\partial x^i$, we get,

$$(Xf)(p) = \sum a^i(p) \frac{\partial f}{\partial x^i}(p)$$

which shows that Xf is a C^∞ function on U. Thus, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map

$$C^\infty(U) \to C^\infty(U)$$
$$f \mapsto Xf$$

Proposition 2.2.1. (Leibniz rule for a vector field). If X is a C^∞ vector field and f and g are C^∞ functions on an open subset U of $\mathbb{R}^n$, then $X(fg)$ satisfies the product rule (Leibniz rule):

$$X(fg) = (Xf)g + fXg.$$

Proof. At each point $p \in U$, the vector X_p satisfies the Leibniz rule:

$$X_p(fg) = (X_pf)g(p) + f(p)X_pg.$$

As p varies over U, this becomes an equality of functions:

$$X(fg) = (Xf)g + fXg.$$

2.3 The Exterior Algebra of Multicovectors

Once one admits linear functions on a tangent space, it is but a small step to consider functions of several arguments linear in each argument. These are the *multilinear functions* on a vector space. The determinant of a matrix, viewed as a function of the column vectors of the matrix, is an example of a multilinear function. Among the multilinear functions, certain ones such as the determinant and the cross product have an anti-symmetric or alternating property: they change sign if two arguments are switched. The alternating multilinear functions with k arguments on a vector space are called *multicovectors of degree* k, or k-covectors for short.

2.3.1 Dual Space

If V and W are real vector spaces, we denote by $\mathrm{Hom}(V, W)$ the vector space of all linear maps $f : V \to W$. Define the dual space $V^\vee$ of V to be the vector space of all real-valued linear functions on V :

$$V^\vee = \mathrm{Hom}(V, R).$$

The elements of $V^\vee$ are called **covectors** or 1-covectors on V .
Let $\{e_1, \ldots, e_n\}$ be a basis for V . Then every v in V is uniquely a linear combination $v = \sum v^i e_i$ with $v^i \in \mathbb{R}$. Let $\alpha^i : V \to \mathbb{R}$ be the linear function that picks out the ith coordinate, $\alpha^i(v) = v^i$. Note that α^i is characterized by $\alpha^i(e_j) = \delta^i_j$.

Proposition 2.3.1. The functions $\alpha^1, \ldots, \alpha^n$ forms a basis for $V^\vee$.

Proof. We first prove that $\alpha^1, \ldots, \alpha^n$ span $V^\vee$. If $f \in V^\vee$ and $v = \sum v^i e_i \in V$, then,

$$f(v) = \sum v^i f(e_i) = \sum f(e_i)\alpha^i(v).$$

Hence, $f = \sum f(e_i)\alpha^i$, which shows that $\alpha^1, \ldots, \alpha^n$ span $V^\vee$.
To show linear independence, suppose $\sum c_i\alpha^i = 0$ for some $c_i \in \mathbb{R}$. Applying both sides to the vector e_j gives

$$0 = \sum_i c_i\alpha^i(e_j) = \sum c_i\delta^i_j = c_j, \quad j = 1, ..., n.$$

Hence, $\alpha^1, \ldots, \alpha^n$ are linearly independent.
This basis $\alpha^1, \ldots, \alpha^n$ for $V^\vee$ is said to be dual to the basis $e_1, \ldots, e_n$ for V .

Corollary. The dual space $V^\vee$ of a finite-dimensional vector space V has the same dimension as V.

2.3.2 Multilinear Functions

Fix a positive integer k. A permutation of the set $A = \{1, \ldots, k\}$ is a bijection $\sigma : A \to A$. More concretely, σ may be thought of as a reordering of the list $1, 2, \ldots, k$ from its natural increasing order to a new order $\sigma(1), \sigma(2), \ldots, \sigma(k)$. Let S_k be the group of all permutations of the set $\{1, \ldots, k\}$. A permutation is even or odd depending on whether it is the product of an even or an odd number of transpositions.

Denote by $V^k = V \times \ldots \times V$ the Cartesian product of k copies of a real vector space V . A function $f : V^k \to \mathbb{R}$ is k-linear if it is linear in each of its k arguments: $f(\ldots, av + bw, \ldots) = af(\ldots, v, \ldots) + bf(\ldots, w, \ldots)$ for all $a, b \in \mathbb{R}$ and $v, w \in V$.

A k-linear function on V is also called a k-tensor on V . We will denote the vector space of all k-tensors on V by $L_k(V)$. If f is a k-tensor on V , we also call k the degree of f .

Definition 2.3.1. A k-linear function $f : V^k \to \mathbb{R}$ is symmetric if

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = f(v_1, \ldots, v_k)$$

for all permutations $\sigma \in S_k$; it is alternating if for all $\sigma \in S_k$,

$$f\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) = (\operatorname{sgn} \sigma) f(v_1, \ldots, v_k)$$

Note. For any two linear functions $f, g : V \to \mathbb{R}$ on a vector space V , the function $f \wedge g : V \times V \to \mathbb{R}$ defined by

$$(f \wedge g)(u, v) = f(u)g(v) - f(v)g(u)$$

is alternating. This is a special case of the wedge product, which we will soon define.

We denote $A_k(V)$ as the *space of all alternating k-linear functions* on a vector space V for $k > 0$. These are also called **alternating k-tensors,** **k-covectors**, or **multicovectors of degree** k on V . For $k = 0$, we define a 0-covector to be a constant, so that $A_0(V)$ is the vector space $\mathbb{R}$. A 1-covector is simply a covector.

2.3.3 Permutation Action on Multilinear Functions

If f is a k-linear function on a vector space V and σ is a permutation in S_k , we define a new k-linear function σf by

$$(\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Thus, f is symmetric if and only if $\sigma f = f$ for all $\sigma \in S_k$ and f is alternating if and only if $\sigma f = (\text{sgn } \sigma) f$ for all $\sigma \in S_k$.

Lemma 2.3.2. If $\sigma, \tau \in S_k$ and f is a k-linear function on V , then $\tau(\sigma f) = (\tau\sigma) f$.

Proof . For $v_1, \ldots, v_k \in V$,

$$\begin{aligned}
\tau(\sigma f)(v_1, \ldots, v_k) &= (\sigma f)(v_{\tau(1)}, \ldots, v_{\tau(k)}) \\
&= (\sigma f)(w_1, \ldots, w_k) \quad (\text{letting } w_i = v_{\tau(i)}) \\
&= f(w_{\sigma(1)}, \ldots, w_{\sigma(k)}) \\
&= f(v_{\tau(\sigma(1))}, \ldots, v_{\tau(\sigma(k))}) \\
&= f(v_{(\tau\sigma)(1)}, \ldots, v_{(\tau\sigma)(k)}) \\
&= (\tau\sigma) f(v_1, \ldots, v_k).
\end{aligned}$$

□

2.3.4 Tensor Product

Let f be a k-linear function and g an l-linear function on a vector space V . Their *tensor product* is the $(k+l)$-linear function $f \otimes g$ defined by

$$(f \otimes g)(v_1, \ldots, v_{k+l}) = f(v_1, \ldots, v_k) g(v_{k+1}, \ldots, v_{k+l})$$

Bilinear Maps : Let $e_1, \ldots, e_n$ be a basis for a vector space V. $\alpha^1, \ldots, \alpha^n$ the dual basis in $V^\vee$, and $<,>: V \times V \to \mathbb{R}$ a bilinear map on V . Set $g_{ij} = < e_i, e_j > \in \mathbb{R}$. If $v = \sum v^i e_i$ and $w = \sum w^i e_i$, then $v^i = \alpha^i(v)$ and $w^j = \alpha^j(w)$. By bilinearity, we can express $<,>$ in terms of the tensor product:

$$< v, w > = \sum v^i w^j < e_i, e^j > = \sum \alpha^i(v) \alpha^j(w) g_{ij} = \sum g_{ij} (\alpha^i \otimes \alpha^j)(v, w).$$

Hence, $<,> = \sum g_{ij} \alpha^i \otimes \alpha^j$.

2.3.5 Wedge Product

If two multilinear functions f and g on a vector space V are alternating, then we would like to have a product that is alternating as well. This motivates the definition of the wedge product, also called the exterior product. For $f \in A_k(V)$ and $g \in A_l(V)$,

$$f \wedge g = \frac{1}{k! l!} A(f \otimes g)$$

explicitly,

$$(f\wedge g)(v_1,\ldots,v_{k+l}) = \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}} f(v_{\sigma(1)},\ldots,v_{\sigma(k)})g(v_{\sigma(k+1)},\ldots,v_{\sigma(k+l)})$$

$f\wedge g$ is alternating. $f\wedge g$ is called the *wedge product* of f and g.

Proposition 2.3.3. The wedge product is anti-commutative : if $f \in A_k(V)$ and $g \in A_l(V)$, then

$$f\wedge g = (-1)^{kl} g\wedge f$$

In fact, $A(f \otimes g) = (\text{sgn } \tau)\ A(g \otimes f)$. Dividing both sides by $k!l!$ we get our result.

Corollary. If f is a multicovector of odd degree on V , then $f\wedge f = 0$.

Lemma 2.3.4. The wedge product is associative. Let V be a real vector space and f, g, h alternating multilinear functions on V of degrees k, l, m, respectively. Then,

$$(f\wedge g)\wedge h = f\wedge(g\wedge h).$$

Under this hypothesis, we also have :

$$f\wedge g\wedge h = \frac{1}{k!l!m!}A(f\otimes g\otimes h).$$

2.3.6 A Basis for k-Covectors

Let $e_1,\ldots,e_n$ be a basis for a real vector space V , and let $\alpha^1,\ldots,\alpha^n$ be the dual basis for $V^\vee$. Introduce the multi-index notation $I = (i_1,\ldots,i_k)$ and write e_I for $(e_{i_1},\ldots,e_{i_k})$ and α^I for $\alpha^{i_1}\wedge\ldots\wedge\alpha^{i_k}$. The alternating k linear functions, α^I , $I = (i_1,\ldots,i_k)$ form a basis for the space $A_k(V)$ of alternating k-linear functions on V.

Lemma 2.3.5. Let $e_1,\ldots,e_n$ be a basis for a vector space V and let $\alpha^1,\ldots,\alpha^n$ be its dual basis in $V^\vee$. If $I = (1 \le i_1 \le \ldots \le i_k \le n)$ and $J = (1 \le j_1 \le \ldots \le j_k \le n)$ are strictly ascending multi-indices of length k, then $\alpha^I(e_J) = \delta^I_J$.

2.3.7 Differential Forms on $\mathbb{R}^n$:

Just as a vector field assigns a tangent vector to each point of an open subset U of $\mathbb{R}^n$, so dually a differential k-form assigns a k-covector on the tangent space to each point of U. The wedge product of differential forms is defined pointwise as the wedge product of multicovectors. Since differential forms exist on an open set, not just at a single point, there is a notion of differentiation for differential forms. In fact, there is a unique one, called the *exterior derivative* (defined later), characterized by three natural properties.

The cotangent space to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$ is defined to be the dual space $(T_p\mathbb{R}^n)^\vee$ of the tangent space $T_p\mathbb{R}^n$. Thus an element of the cotangent space $T_p^*(\mathbb{R}^n)$ is a covector or a linear functional on the tangent space $T_p(\mathbb{R}^n)$. A covector field or a differential 1-form on an open subset U of $\mathbb{R}^n$ is a function ω that assigns to each point p in U a covector $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\omega : U \to \bigcup_{p\in U} T_p^*(\mathbb{R}^n),$$

$$p \mapsto \omega_p \in T_p^*(\mathbb{R}^n).$$

Note that in the union $p \in U$, $T_p^*(\mathbb{R}^n)$, the sets $T_p^*(\mathbb{R}^n)$ are all disjoint. We call a differential 1-form a 1-form for short. From any C^∞ function $f : U \to \mathbb{R}$, we can construct a 1-form df , called the differential of f , as follows. For $p \in U$ and $X_p \in T_pU$, define

$$(df)_p(X_p) = X_pf$$

The directional derivative of a function in the direction of a tangent vector at a point p sets up a bilinear pairing

$$T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$$
$$(X_p, f) \mapsto \;< X_p, f > \;=\; X_pf.$$

One may think of a tangent vector as a function on the second argument of this pairing: $< X_p \;,\; \cdot >$. The differential $(df)_p$ at p is a function on the first argument of the pairing: $(df)_p = < \cdot \;,\; f >$.

Proposition 2.3.6. If $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$, $\{(dx^1)_p, \ldots, (dx^n)_p\}$ is the basis for the cotangent space $T_p^*(\mathbb{R}^n)$ dual to the basis $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\}$ for the tangent space $T_p(\mathbb{R}^n)$.

Proof. By definition,

$$(dx^i)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p x^i = \delta^i_j.$$

If ω is a 1-form on an open subset U of $\mathbb{R}^n$, then at each point $p \in U$, ω can be written as a linear combination:

$$\omega_p = \sum a_i(p)(dx^i)_p \quad \text{for some} \;\; a_i(p) \in \mathbb{R}$$

The covector field ω is said to be C^∞ on U if the coefficient functions a_i are all C^∞ on U.

Proposition 2.3.7. (The Differential in terms of coordinates) If $f : U \to \mathbb{R}$ is a C^∞

function on an open set U in $\mathbb{R}^n$, then

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Proof. At each point $p \in U$,

$$(df)_p = \sum a_i(p)(dx^i)_p$$

for some real numbers $a_i(p)$ depending on p. Thus $df = \sum a_i dx^i$ for some real functions a_i on U. Applying both sides to the coordinate vector field $\dfrac{\partial}{\partial x^j}$:

$$df\left(\frac{\partial}{\partial x^j}\right) = \sum_i a_i dx^i \left(\frac{\partial}{\partial x^j}\right) = \sum_i a_i \delta^i_j = a_j$$

We know,

$$df\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}.$$

which proves the proposition.

2.3.8 Differential k-Forms

A differential form ω of degree k or a k-form on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U an alternating k-linear function on the tangent space $T_p(\mathbb{R}^n)$, i.e $\omega_p \in A_k(T_p\mathbb{R}^n)$. Since , $A_1(T_p\mathbb{R}^n) = T^*_p(\mathbb{R}^n)$, the definition of a k-form generalizes that of a 1-form.
A basis of $A_k(T_p\mathbb{R}^n)$ is :

$$dx^I_p = dx^{i_1}_p \wedge \ldots \wedge dx^{i_k}_p \ , \quad 1 \leq i_1 \leq \ldots \leq i_k \leq n$$

Therefore, at each point p in U, ω_p is a linear combination $\omega_p = \sum a_I(p) dx^I_p$, $1 \leq i_1 \leq \ldots \leq i_k \leq n$, and a k-form ω on U is a linear combination, $\omega = \sum a_I dx^I$, with function coefficients $a_I : U \to \mathbb{R}$. We say that a k-form ω is C^∞ on U if all the coefficients a_I are C^∞ functions on U.

Denote by $\Omega^k(U)$ the vector space of C^∞ k-forms on U. A 0-form on U assigns to each point p in U an element of $A_0(T_p(\mathbb{R}^n) = \mathbb{R}$. Thus a 0-form on U is simply a function on U and $\Omega^0(U) = C^\infty(U)$.

The wedge product of a k-form ω and an l-form τ on an open set U is defined pointwise:

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p \ , \quad p \in U$$

In term of coordinates, if $\omega = \sum_I a_I dx^I$ and $\tau = \sum_J b_J dx^J$, then

$$\omega \wedge \tau = \sum_{I,J \text{ disjoint}} (a_I b_J) dx^I \wedge dx^J. \quad \text{(wedge product is anti commutative)}$$

The wedge product is a bilinear map

$$\wedge : \Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$$

Remarks. In case one of the factors has degree 0, say $k = 0$, the wedge product is the pointwise multiplication of a C^∞ l-form by a C^∞ function:

$$(f \wedge \omega)_p = f(p) \wedge \omega_p = f(p)\omega_p$$

Differential Forms as Multilinear Functions on Vector Fields : If ω is a C^∞ 1-form and X is a C^∞ vector field on an open set U in $\mathbb{R}^n$, we define a function $\omega(X)$ on U by the formula :

$$\omega(X)_p = \omega_p(X_p), \quad p \in U$$

Written out in coordinate,

$$\omega = \sum a_i dx^i \ , \quad X = \sum b^j \frac{\partial}{\partial x^j} \qquad \text{for some } a_i, b^j \in C^\infty(U)$$

2.3.9 The Exterior Derivative

To define the exterior derivative of a C^∞ k-form on an open subset U of $\mathbb{R}^n$, we first define the 0-form : the exterior derivative of a C^∞ function $f \in C^\infty(U)$ is defined to be its differential $df \in \Omega^1(U)$, in terms of coordinates

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Definition 2.3.2. For $k \geq 1$, if $\omega = \sum_I a_I dx^I \in \Omega^k(U)$, then

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \in \Omega^{k+1}(U).$$

Example. Let ω be the 1-form $f dx + g dy$ on $\mathbb{R}^2$, where f and g are $\mathbb{C}^\infty$ functions on $\mathbb{R}^2$. To simplify the notation, write $f_x = {}^{\partial f}/_{\partial x}$, $f_y = {}^{\partial f}/_{\partial y}$. Then,

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy \\ &= (g_x - f_y) dx \wedge dy \end{aligned}$$

In this computation $dy \wedge dx = -dx \wedge dy$ and $dx \wedge dx = dy \wedge dy = 0$ by the anti commutative property of wedge product.

Proposition 2.3.8 :

(i) The exterior differentiation $d : \Omega^*(U) \to \Omega^*(U)$ is an anti derivation of degree 1 :

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg\, \omega} \omega \wedge d\tau.$$

(ii) $d^2 = 0$.

Proof. (i) For $\omega = f\ dx^I$ and $\tau = g\ dx^J$,

$$\begin{aligned} d(\omega \wedge \tau) &= d(fg dx^I \wedge dx^J) \\ &= \sum \frac{\partial (fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum \frac{\partial f}{\partial x^i} g dx^i \wedge dx^I \wedge dx^J + \sum f \frac{\partial g}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum \frac{\partial f}{\partial x^i} g dx^i \wedge dx^I \wedge dx^J + (-1)^k \sum f dx^I \wedge \frac{\partial g}{\partial x^i} dx^i \wedge dx^J \\ &= d\omega \wedge \tau + (-1)^k \omega \wedge d\tau \end{aligned}$$

(ii) By the $\mathbb{R}$ linearity of d , it suffices to show that $d^2\omega = 0$ for $\omega = f dx^I$. We compute,

$$d^2(f dx^I) = d\left(\sum \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) = \sum \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I$$

In this sum, if $i = j$, then $dx^j \wedge dx^i = 0$, if $i \neq j$, then $\dfrac{\partial^2 f}{\partial x^i \partial x^j}$ is symmetric in i and j but $dx^j \wedge dx^i$ is alternating in i and j, so the terms with $i \neq j$ pair up and cancel each other. Therefore , $d^2(f dx^I) = 0$.

Chapter 3

Manifolds

Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. Many concepts from $\mathbb{R}^n$, such as differentiability, point-derivations, tangent spaces, and differential forms, carry over to a manifold.

3.1 What is a Manifold?

We first recall a few definitions from point-set topology. A topological space is *second countable* if it has a countable basis. A *neighborhood* of a point p in a topological space M is any open set containing p. An *open cover* of M is a collection $\{U_\alpha\}_{\alpha \in A}$ of open sets in M whose union $\bigcup_{\alpha \in A} U_\alpha$ is M. A topological space M is *locally Euclidean* of dimension n if every point p in M has a neighborhood U such that there is a *homeomorphism* ϕ from U onto an open subset of $\mathbb{R}^n$. A function between two topological space is a homeomorphism if it is a bijection, it is continuous and the inverse function is continuous.

Definition 3.1.1. A topological **manifold** is a Hausdorff, second countable, locally Euclidean space. Manifold is of dimension n if it is locally Euclidean of dim n.
Here, $(U\ ,\ \phi : U \to \mathbb{R}^n)$ is called a *chart*. U is called *coordinate neighborhood* or *coordinate open set*. ϕ is called *coordinate map or system* on U. A chart $\big(U, \phi\big)$ is *centered* at $p \in U$ if $\phi(p) = 0$.

Example: (1) The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbf{1}_{\mathbb{R}^n})$, where $\mathbf{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold, with chart $(U, \mathbf{1_U})$.

(2) The unit circle $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a 1-dimensional topological manifold. Indeed the map $\mathbb{R} \to S^1$ given by $\theta \mapsto (\cos\theta, \sin\theta)$ has restrictions to small ope sets which are homeomorphisms. More generally the n-sphere $S^n := \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum(x^i)^2 = 1\}$ is a topological manifold.

Definition 3.1.2. (compatible charts)
Suppose $\big(U\ ,\ \phi : U \to \mathbb{R}^n\big)$ and $\big(V\ ,\ \psi : U \to \mathbb{R}^n\big)$ are two charts on a topological manifold. $U \cap V$ is open in U. $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. $\phi(U \cap V)$ open in $\mathbb{R}^n$ and $\psi(U \cap V)$ open in $\mathbb{R}^n$.

Definition 3.1.3. Two charts $\big(U\ ,\ \phi : U \to \mathbb{R}^n\big)$ and $\big(V\ ,\ \psi : U \to \mathbb{R}^n\big)$ are C^∞ **compatible** if the two maps :

$$\begin{aligned} \phi \circ \psi^{-1} &: \psi(U \cap V) \longrightarrow \phi(U \cap V) \\ \psi \circ \phi^{-1} &: \phi(U \cap V) \longrightarrow \psi(U \cap V) \end{aligned}$$

are C^∞ maps.

1. $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are called **transition functions** between the charts.
2. If $U \cap V$ is empty, they are automatically C^∞ compatible.

Definition 3.1.4. A C^∞ *-Atlas* on a locally Euclidean space M is a collection $U = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞ compatible charts that cover M i.e, $M = \bigcup\limits_\alpha U_\alpha$.

Definition 3.1.5. A chart (V, ψ) is *compatible with an atlas* $\{(U_\alpha, \phi_\alpha)\}$ if it is compatible with all the charts (U_α, ϕ_α) of the atlas.

3.1.1 Product Manifold :

Let M and N be C^∞ manifolds, then $M \times N$ with its product topology is Hausdorff and second countable. If $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N dimensions m and n respectively, then the collection :

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

$$\begin{aligned} &\phi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n) \\ &\phi_\alpha \times \psi_i(u, v) \ = \ (\phi_\alpha(u), \psi_i(v)) \end{aligned}$$

Since each of the components are continuous, hence so the product function is continuous. Similarly,

$$(\phi_\alpha \times \psi_i)^{-1}(u, v) \ = \ (\phi_\alpha^{-1}(u), \psi_i^{-1}(v))$$

Since ϕ_α^{-1} and ψ_i^{-1} are continuous functions hence $(\phi_\alpha \times \psi_i)^{-1}$ is also continuous.

Since $M \times N \times P = (M \times N) \times P$ is the successive product of pairs of spaces, if M, N and P are manifolds, then so is $M \times N \times P$. This notion can be generalized to product of n manifolds. For example : the n dimensional torus $S^1 \times \ldots \times S^1$ (n times) is a product manifold.

3.2 Smooth Manifold

The first requirement for transferring the ideas of calculus to manifolds is some notion of "smoothness." For the simple examples of manifolds we described above, all subsets of Euclidean spaces, it is fairly easy to describe the meaning of smoothness on an intuitive level. For example, we might want to call a curve "smooth" if it has a tangent line that varies continuously from point to point, and similarly a "smooth surface" should be one that has a tangent plane that varies continuously from point to point. But for more sophisticated applications, it is an undue restriction to require smooth manifolds to be subsets of some ambient Euclidean space. We will think of a smooth manifold as a set with two layers of structure: first a topology, then a smooth structure.

Recall, given any arbitrary subset $X \subseteq \mathbb{R}^m$, a function $f : X \to \mathbb{R}^n$ is called *smooth function* if every point in X has some neighborhood where f can be extended to a smooth function.

Definition 3.2.1. (1st definition) A manifold is **smooth manifold** if the transition maps are smooth.

An atlas $\mathcal{A}$ on a locally Euclidean space is said to be *maximal* if it is not contained in a larger atlas i.e, if $\mathcal{B}$ is another atlas containing $\mathcal{A}$ then $\mathcal{B} = \mathcal{A}$.

Definition 3.2.2. (2nd definition) **Smooth or C^∞ manifold** is a topological manifold M together with a maximal atlas. The maximal atlas is also called a differentiable structure on M.

Recall that, Given two manifolds M and N, a differentiable map $f : M \to N$ is called a *diffeomorphism* if it is a bijection and its inverse $f^{-1} : N \to M$ is differentiable as well.

A function $f : M \to N$ is a map between topological manifolds if f is continuous. It is a smooth map of smooth manifolds M, N if for any smooth charts (U, ϕ) of M and (V, ψ) of N, the function :

$$\psi \circ f \circ \phi^{-1} \ : \ \phi \left(U \cap f^{-1}(V)\right) \ \longrightarrow \ \psi(V)$$

is a C^∞ diffeomorphism.

Remarks : C^∞ compatibility of charts is reflexive , symmetric but not transitive. Because, if (U_1, ϕ_1) is compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is compatible with (U_3, ϕ_3), then (U_1, ϕ_1) is compatible with (U_3, ϕ_3) on $U_1 \cap U_2 \cap U_3$ only, not on $U_1 \cap U_3$ because the transition maps are defined on (U_3, ϕ_3) on $U_1 \cap U_2 \cap U_3$ as : $\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$

3.2.1 Smooth Maps on a Manifold :

Definition 3.2.3. Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be a C^∞ or *smooth* at a point $p \in M$ if there is a chart (U, ϕ) about $p \in M$ such that , $f \circ \phi^{-1} \ : \ \phi(U) \to \mathbb{R}^n$ is C^∞ at $\phi(p)$. f is said to be C^∞ on M if f is C^∞ at every point of M.

Proposition 3.2.1. Definition of smoothness of a function f at a point is independent of the chart (U, ϕ).

Proof: If $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, ψ) be any other chart about $p \in M$, then on $\psi(V \cap U)$

$$f \circ \psi^{-1} = \ (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$$

is also C^∞ at $\phi(p)$.

3.2.2 Smoothness of a Real Valued Function :

Let M be a manifold of dimension n and $f : M \to \mathbb{R}$ is a real valued function on M. The following are equivalent :

(i) $f : M \to \mathbb{R}$ is C^∞.

(ii) M has an atlas such that or every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.

(iii) For every chart (V, ψ) on M, $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞ on $\psi(V)$.

Proof : (ii) $\implies$ (i) By (ii) every point $p \in M$ has a coordinate neighborhood (U, ϕ) such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$. So f is C^∞ map.
(i) $\implies$ (iii) Let (V, ψ) be a chart on M. Let $p \in V$. $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$.
Because f smooth implies $\exists$ a chart (U, ϕ) about p st $f \circ \phi^1$ is C^∞ at $\phi(p)$ and if $V, \psi)$ is any other chart about p in M then on $\psi(U \cap V)$, $f \circ \psi^{-1} = \ (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$ is C^∞ at $\psi(p)$.
Since, p is an arbitrary point in V, $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞ on $\psi(V)$.
(iii) $\implies$ (i) obvious.

3.2.3 Diffeomorphism :

Definition 3.2.4. A diffeomorphism of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞.

Proposition 3.2.2. If (U, ϕ) is a chart on manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Proof : To check, ϕ and ϕ^{-1} are both smooth. Consider atlas $\{(U, \phi)\}$ on U and atlas $\{(\phi(U), \mathbf{1}_{\phi(U)})\}$ with single chart on $\phi(U)$. $\mathbf{1}_{\phi(U)} \circ \phi \circ \phi^{-1} : \ \phi(U) \to \ \phi(U)$ is the identity map is C^∞. To test the smoothness of $\phi^{-1} : \phi(U) \to U$,
since, $\phi \circ \phi^{-1} \circ \mathbf{1}_{\phi(U)} = \mathbf{1}_{\phi(U)} : \phi(U) \to \phi(U)$, $\quad \therefore \phi^{-1}$ is also C^∞.

Remark : Let N be a manifold. A vector valued function $F : N \to \mathbb{R}^m$ is C^∞ iff its component functions $F^1, \ldots F^m$ where $F^i : N \to \mathbb{R}$ are all C^∞ $\forall i = 1(1)m$.

Example : (Smoothness of a projection map). Let M and N be manifolds and $\pi : M \times N \to M \;\; \pi(p, q) = p$ the projection to the first factor. Prove that π is a C^∞ map.

Proof : Let (p, q) be an arbitrary point of $M \times N$. Suppose $(U, \phi) = (U, x_1, \dots, x_m)$ and $(V, \psi) = (V, y_1, \dots, y_n)$ are coordinate neighborhoods of p and q in M and N respectively. Hence,

$$(U \times V, \phi \times \psi) = (U \times V, x_1, \dots, x_m, y_1, \dots, y_n)$$

which is a C^∞ map from $(\phi \times \psi)(U \times V)$ in $\mathbb{R}^{m+n}$ to $\phi(U)$ in $\mathbb{R}^m$, so π is C^∞ at (p, q). Since (p, q) was an arbitrary point in $M \times N$, π is C^∞ on $M \times N$.

Example. Let M_1, M_2 and N be manifolds of dimensions m_1, m_2 and n respectively. Prove that a map $(ff_1, f_2) : N \to M_1 \times M_2$ is C^∞ iff $f_i : N \to M_i$, $i = 1, 2$ are both C^∞.

Proof : $\Longrightarrow$ if (f_1, f_2) is C^∞ , $(f_1, f_2)(n) = (f_1(n), f_2(n))$ which means by projection mapping, both f_1 and f_2 are C^∞ maps.
$\Longleftarrow$ if we have $f_i : N \to M_i$, $i = 1, 2$ are both C^∞ maps, then we can consider the product topology on $M_1 \times M_2$

3.2.4 Partial Derivatives :

On a manifold M of dimension n , let (U, ϕ) be a chart and f a C^∞ function. As a function into $\mathbb{R}^n$, ϕ has n components $x^1, \dots, x^n$. If $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then $x^i = r^i \circ \phi$.
For, $p \in U$, we define the **partial derivative** $\partial f/\partial x^i$ of f with respect to x^i at p to be :

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)) := \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1})$$

Since $p = \phi^{-1}(\phi(p))$, this equation may be written in the form

$$\frac{\partial f}{\partial x^i}(\phi^{-1}(\phi(p))) \;=\; \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

Thus as functions on $\phi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} \;=\; \frac{\partial (f \circ \phi^{-1})}{\partial r^i}.$$

The partial derivative $\partial f/\partial x^i$ is C^∞ on U because its pullback $(\partial f/\partial x^i) \circ \phi^{-1}$ is $\mathbb{C}^{-1}$ on $\phi(U)$.

Proposition 3.2.3. Suppose $(U, x^1, \dots, x^n)$ is a chart on a manifold. Then $\partial xi/\partial x^j = \delta^i_j$.

Proof : At a point $p \in U$, by the definition of ${\partial}/{\partial x^j}|_p$,

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial(x^i \circ \phi^{-1}}{\partial r^j}(\phi(p)) \;=\; \frac{\partial(r^i \circ \phi \circ \phi^{-1})}{\partial r^j}(\phi(p)) \;=\; \frac{\partial r^i}{\partial r^j}(\phi(p)) \;=\; \delta^i_j \quad \square$$

Definition 3.2.5. Let $F : N \to M$ be a smooth map , and let $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by :

$$F^i := y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

is the i th component of F in the chart (V, ψ). Then the matrix $[{\partial F^i}/{\partial x^j}]$ is called the **Jacobian matrix** of F relative to the charts (U, ϕ) and (V, ψ).
If M and N has same dimensions then the determinant $\det[{\partial F^i}/{\partial x^j}] \;=\; {\partial(F^1,\ldots,F^n)}/{\partial(x^1,\ldots,x^n)}$ is called the *Jacobian determinant* of F relative to the two charts.

Jacobian Matrix of a Transition Map :

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be overlapping charts on a manifold M. The transition map $\phi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^{\ltimes}$. It's Jacobian matrix $J(\psi \circ \phi^{-1})$ at $\phi(p)$ is the matrix $[{\partial y^i}/{\partial x^j}]$ of partial derivatives at p.

Proof : By definition, $J(\psi \circ \phi^{-1} = [{\partial(\psi \circ \phi^{-1})^i}/{\partial r^j}]$, where

$$\frac{\partial(\psi \circ \phi^{-1})^i}{\partial r^j}(\phi(p)) \;=\; \frac{\partial(r^i \circ \psi \circ \phi^{-1})}{\partial r^j}(\phi(p)) \;=\; \frac{\partial(y^i \circ \phi^{-1})}{\partial r^j}(\phi(p)) \;=\; \frac{\partial y^i}{\partial x^j}(p).$$

3.2.5 The Inverse Function Theorem :

A diffeomorphism $F : U \to F(U) \subset \mathbb{R}^n$ of an open subset U of a manifold may be thought of as a coordinate system on U. We say that a C^∞ map $F : N \to M$ is a **locally invertible** or a local diffeomorphism at $p \in N$ if p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism.

Given n smooth functions $F^1, \ldots, F^n$ in a neighborhood of a point p in a manifold N of dimension n, question arises whether they would form a coordinate system, possibly on a smaller neighborhood of p, which is equivalent to asking whether $F = (F^1, \ldots, F^n) : N \to \mathbb{R}^n$ is a local diffeomorphism at p. The inverse function theorem provides an answer.

Theorem 3.2.4. (Inverse function theorem for manifolds)
Let $F : N \to M$ be a C^∞ map between two manifolds of the same dimension and $p \in N$. Suppose for some charts $(U, \phi) = (U, x^1, \ldots, x^n)$ about $p \in N$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ about $F(p)$ in M, $F(U) \subset V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p iff its Jacobian determinant $\det[{\partial F^i}/{\partial x^j(p)}]$ is non zero.

Proof : Since $F^i = y^i \circ F = r^i \circ \psi \circ F$, the Jacobian matrix of F relative to the charts (U, ϕ) and (V, ψ) is :

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial (r^i \circ \psi \circ F)}{\partial x^j}(p)\right] = \left[\frac{\partial (r^i \circ \psi \circ F \circ \phi^{-1})}{\partial r^j}(\phi(p))\right],$$

which is precisely the Jacobian matrix at $\phi(p)$ of the map

$$det\left[\frac{\partial F^i}{\partial x^j}(p)\right] = det\left[\frac{\partial r^i \circ (\psi \circ F \circ \phi^{-1})}{\partial r^j}(\phi(p))\right] \neq 0$$

iff $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$. Since, ϕ and ψ are diffeomorphisms , this proves the local invertibility of F at p.

Corollary 3.2.5. Let N be a manifold of dimension n. A set of n smooth function $F^1, \ldots, F^n$ defined on a coordinate neighborhood $(U, x^1, \ldots, x^n)$ of a point $p \in N$ forms a coordinate system about p iff the Jacobian determinant det $[\partial F^i/\partial x^j(p)]$ is non zero.

Proof : Let $F = (F^1, \ldots, F^n) : U \to \mathbb{R}^n$. Then, $\det[\partial F^i/\partial x^j(p)] \neq 0$
$\iff$ $F : U \to \mathbb{R}^n$ is locally invertible at p (by the inverse function theorem)
$\iff$ there is a neighborhood W of p in N such that : $W \to F(W)$ is a diffeomorphism (by the definition of local invertibility)
$\iff$ $(W, F^1, \ldots, F^n)$ is a coordinate chart about p in the differentiable structure of N.

Example : Find all the points in $\mathbb{R}^2$ in a neighborhood of which the functions $x^3 + y^3, y$ can serve as a local coordinate system.

Solution : Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = (x^3 + y^3, y)$$

The map F can serve as a coordinate map in a neighborhood of p iff it is a local diffeomorphism at p. The Jacobian determinant of F is :

$$\frac{\partial(F^1, F^2)}{\partial(x, y)} = det\begin{bmatrix} 3x^2 & 3y^2 \\ 0 & 1 \end{bmatrix} = 3x^2$$

By the inverse function theorem, F is a local diffeomorphism at $p = (x, y)$ if and only if $x \neq 0$. Thus, F can serve as a coordinate system at any point p not on the y-axis.

Chapter 4

Tangent Space of Manifold

4.1 Tangent Space

A basic principle in manifold theory is the linearization principle, according to which a manifold can be approximated near a point by its tangent space at the point, and a smooth map can be approximated by the differential of the map. Hence the tangent space to a manifold at a point is the vector space of derivations at the point. The collection of tangent spaces to a manifold can be given the structure of a vector bundle; it is then called the tangent bundle of the manifold. Intuitively, a vector bundle over a manifold is a locally trivial family of vector spaces parametrized by points of the manifold.

Definition 4.1.1. We define a **germ** of a C^∞ function at p in M is the equivalent class of C^∞ functions defined on a neighborhood of p in M, two such functions are said to be equivalent if they agree on a smaller neighborhood of p.
The set of germs of C^∞ valued functions at p in M is denoted by $C_p^\infty(M)$, which forms an algebra over $\mathbb{R}$.

Definition 4.1.2. A derivation at a point in a manifold M or a **point derivation** of $C_p^\infty(M)$ is a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that :

$$D(fg) = (Df)g(p) + f(p)Dg$$

Definition 4.1.3. A **tangent vector** at a point p in a manifold M is a derivation at p.
The tangent vectors at p form a vector space $T_p(M)$, called the tangent space of M at p.

Remarks : (Tangent space to an open subset) If U is an open set containing p in M, then the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$. Hence, $T_pU = T_pM$.

4.1.1 Transition Matrix for Coordinate Vectors :

Suppose $(U, x_1, \ldots, x_n)$ and $(V, y_1, \ldots, y_n)$ are two coordinate charts on M. Then on $U \cap V$:

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i}$$

Proof : At each point $p \in U \cap V$, the sets $\left\{ \frac{\partial}{\partial x^j}\bigg|_p \right\}$ and $\left\{ \frac{\partial}{\partial y^i}\bigg|_p \right\}$ are both bases for the tangent space T_pM so there is a matrix $[a^i_j(p)]$ of real numbers such that on $U \cap V$,

$$\frac{\partial}{\partial x^j} = \sum_k a^k_j \frac{\partial}{\partial y^k}$$

Applying both sides of the equation to y^i , we get

$$\begin{aligned}\frac{\partial y^i}{\partial x^j} &= \sum_k a^k_j \frac{\partial y^i}{\partial y^k} \\ &= \sum_k a^k_j \delta^i_k \\ &= a^i_j\end{aligned}$$

□

4.2 Differential of a Map :

Let $F : N \to M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map F induces a linear map of tangent spaces called its differential at p,

$$F_* : T_pN \to T_{F(p)}M$$

If $X_p \in T_pN$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined as :

$$(F_*(X_p))(f) = X_p(f \circ F) \in \mathbb{R} \quad \text{for } f \in C^\infty_{F(p)}(M)$$

f is a germ at $F(p)$, represented by a C^∞ function in a neighborhood of $F(p)$.

Proposition 4.2.1. $F_*(X_p)$ is a derivation at $F(p)$ and $F_* : T_pN \to T_{F(p)}M$ is a linear map.

Differential of a Map between Euclidean Spaces :

Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and p is a point in $\mathbb{R}^n$. Let $x^1, \ldots, x^n$ be the coordinates on $\mathbb{R}^n$ and $y^1, \ldots, y^m$ be the coordinates on $\mathbb{R}^m$. Then the tangent vectors $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ form a basis for the tangent space $T_p(\mathbb{R}^n$ and $\partial/\partial y^1|_{F(p)}, \ldots, \partial/\partial y^m|_{F(p)}$ form a basis for the tangent space $T_F(p)(\mathbb{R}^m$.

The linear map $F_* : T_p(\mathbb{R}^n \to T_F(p)(\mathbb{R}^m$ is described by the matrix $[a^i_j]$ relative to these two bases :

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) \;=\; \sum_k a_j^k \frac{\partial}{\partial y^k}\bigg|_{F(p)} \quad , \quad a_j^k \in \mathbb{R}$$

Let $F^i = y^i \circ F$ be the i th component of F. We can find a_j^i by evaluating the RHS and LHS on y^i :

$$\begin{aligned} RHS = \sum_k \frac{\partial}{\partial y^k}\bigg|_{F(p)} y^i \;&=\; \sum_k a_j^k \delta_k^i \;=\; a_j^i \\ LHS = F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) y^i \;&=\; \frac{\partial}{\partial x^j}\bigg|_p \left(y^i \circ F\right) \;=\; \frac{\partial F^i}{\partial x^j}(p) \end{aligned}$$

So the matrix F_* relative to the bases $\left\{\frac{\partial}{\partial x^j}\bigg|_p\right\}$ and $\left\{\frac{\partial}{\partial y^i}\bigg|_{F(p)}\right\}$ is $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$.

This is precisely the Jacobian matrix of the derivative of F at p. Thus, the differential of a map between manifolds generalizes the derivative of a map between Euclidean spaces.

4.2.1 The Chain Rule :

Let $F : N \to M$ and $G : M \to P$ be smooth maps of manifolds and $p \in N$. The differentials of F at p and G at $F(p)$ are linear maps

$$T_pN \xrightarrow{F_{*,p}} T_{F(p)}M \xrightarrow{G_{x,F(p)}} T_{G(F(p))}P.$$

Theorem 4.2.2. (*The Chain Rule*) If $F : N \to M$ and $G : M \to P$ are smooth map of manifolds and $p \in N$, then ,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$$

Proof : Let $X_p \in T_pN$ and let f be a smooth function at $G(F(p))$ in P. Then,

$$((G \circ F)_* X_p)f \;=\; X_p(f \circ G \circ F)$$

and

$$((G_* \circ F_*)X_p)f \;=\; (G_*(F_*X_p))f \;=\; (F_*X_p)(f \circ G) \;=\; X_p(f \circ G \circ F).$$

Remarks : The differential of the identity map $\mathbb{1}_M : M \to M$ at any point p in M is the identity map

$$\mathbb{1}_{T_pM} : T_pM \to T_pM$$

because,

$$((\mathbb{1}_M)_* X_p)f \ = \ X_p(f \circ \mathbb{1}_M \ ) \ = \ X_p f$$

for any $X_p \in T_pM$ and $f \in \mathbb{C}_p^\infty(M)$.

Corollary 4.2.3. If $F : N \to M$ is a diffeomorphism of manifolds and $p \in N$, then $F_* : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Proof : Since F is a diffeomorphism it has a differentiable inverse $G : M \to N$ such that $G \circ F = \mathbb{1}_N$ and $F \circ G = \mathbb{1}_M$. By chain rule,

$$(G \circ F)_* = G_* \circ F_* = (\mathbb{1}_N)_* = \mathbb{1}_{T_pN}$$
$$(F \circ G)_* = F_* \circ G_* = (\mathbb{1}_M)_* = \mathbb{1}_{T_{F(p)}M} \ .$$

Hence, F_* and G_* are isomorphisms.

Corollary 4.2.4. (*Invariance of dimension*) If an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$.

Proof : Let $F : U \to V$ be a diffeomorphism and let $p \in U$. By previous corollary, $F_{*,p} : T_pU \to T_{F(p)}V$ is an isomorphism of vector spaces. Since there are vector space isomorphisms $T_pU \equiv \mathbb{R}^n$ and $T_{F(p)} \equiv \mathbb{R}^m$, we must have that $n = m$.

4.2.2 Bases for Tangent Spaces at a point

Given a coordinate neighborhood $(U, \phi) = (U, x_1, ..., x_n)$ about a point p in a manifold M, we recall the definition of the partial derivatives $\partial/\partial x^i$. Let $r_1, ..., r_n$ be the standard coordinates on $\mathbb{R}^n$. We set , $x^i = r^i \circ \phi$. Since, $\phi : U \to \mathbb{R}^n$ a diffeomorphism onto its image, the differential,

$$\phi_* : T_pM \to T_{\phi(p)}\mathbb{R}^n$$

is a vector space isomorphism. In particular, the tangent space T_pM has the same dimension as the manifold M.

Proposition 4.2.5. If f is a smooth function in a neighborhood of p. Then,

$$\left.\frac{\partial}{\partial x^i}\right|_p \ = \ \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1}) \ \in \mathbb{R}$$

In other words ,

$$\phi_* \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) \ = \ \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}$$

Proof : For any $f \in \mathbb{C}^\infty_{\phi(p)}(\mathbb{R}^n)$,

$$\begin{aligned}
\phi_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f &= \left.\frac{\partial}{\partial x^i}\right|_p (f \circ \phi) && \text{(definition of } \phi_*) \\
&= \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi \circ \phi^{-1}) && \left(\text{definition of } \left.\frac{\partial}{\partial x^i}\right|_p\right) \\
&= \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f. && \square
\end{aligned}$$

Proposition 4.2.6. If $(U, \phi) = (U, x^1, \ldots, x^n)$ is a chart containing p, The tangent space T_pM has basis

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

Proof : An isomorphism of vector spaces carries basis to basis. The isomorphism $\phi_* : T_pM \to T_{\phi(p)}(\mathbb{R}^n)$ maps $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\}$ to $\{\partial/\partial r^1|_{\phi(p)}, \ldots, \partial/\partial r^n|_{\phi(p)}\}$, which is a basis for the tangent space $T_{\phi(p)}(\mathbb{R}^n)$. Therefore, $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\}$ is a basis for T_pM.

4.2.3 A Local Expression for the Differential :

Given a smooth map $F : N \to M$ o manifolds and $p \in N$, let $(U, x^1, \ldots, x^n)$ be a chart about $p \in N$ and let $(V, y^1, \ldots, y^m)$ be a chart about $F(p)$ in M. Our aim is to find a local expression for the differential $F_{*,p} : T_pN \to T_{F(p)}M$ relative to the two charts such that ,

$$F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{k=1}^{m} a_j^k \left.\frac{\partial}{\partial y^k}\right|_{F(p)}, \quad j = 1(1)n.$$

where,

$$\frac{\partial F^i}{\partial x^j}(p) = a_j^i$$

where $F^i = y^i \circ F$ is the i th component of F. That is the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is represented by the matrix $[\partial F^i/\partial x^j(p)]$.

Example 4.2.7. (The Chain Rule in Calculus notation) Suppose $w = G(x, y, z)$ is a C^∞ function : $\mathbb{R}^3 \to \mathbb{R}$ and $(x, y, z) = F(t)$ is a C^∞ function : $\mathbb{R} \to \mathbb{R}^3$. Under composition,

$$w = (G \circ F)(t) = G(x(t), y(t), z(t))$$

becomes a C^∞ function of $t \in \mathbb{R}$. The differentials F_* , G_*, and $(G \circ F)_*$ are represented by the matrices

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}, \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}, \frac{dw}{dt},$$

respectively. Since composition of linear maps is represented by matrix multiplication , so we get :

$$(G \circ F)_* \;=\; G_* \circ F_*$$

$$\Longrightarrow \frac{dw}{dt} \;=\; \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} \;=\; \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

which is the usual form of chain rule.

4.2.4 Curves in a Manifold :

A *smooth curve* in a manifold M is by definition a smooth map $c : (a, b) \to M$ from some open interval (a, b) into M . Usually we assume $0 \in (a, b)$ and say that c is a curve starting at p if $c(0) = p$. The *velocity vector* $c'(t_0)$ of the curve c , which is the velocity of c at the point $c(t_0)$ at time $t_0 \in (a, b)$ is defined to be

$$c'(t_0) := c_* \left(\left. \frac{d}{dt} \right|_{t_0} \right) \;\in\; T_{c(t_0)}\, M.$$

Notation : Alternative notation of $c'(t_0)$ are $\frac{dc}{dt}(t_0)$ and $\left. \frac{d}{dt} \right|_{t_0} c$.

By our definition, $c'(t)$ is a tangent vector at $c(t)$, hence a multiple of $\left. \frac{d}{dx} \right|_{c(t)}$.

Proposition 4.2.8. (Velocity of a Curve in Local Coordinates). Let $c : (a, b) \to M$ be a smooth curve, and let $(U, x^1, \ldots, x^n)$ be a coordinate chart about $c(t)$. Set $c^i = x^i \circ c$ for the i th component of a c in the chart. Then $c'(t)$ is given by

$$c'(t) \;=\; \sum_{i=1}^{n} c^i(t) \left. \frac{\partial}{\partial x^i} \right|_{c(t)} .$$

Thus, relative to the basis $\{\partial/\partial x^i|_p\}$ for $T_{c(t)}M$, the velocity $c'(t)$ is represented by the column vector

$$\begin{bmatrix} c^1(t) \\ \vdots \\ c^n(t) \end{bmatrix}$$

Thus every smooth curve c at p in a manifold M gives rise to a tangent vector $c'(0)$ in T_pM. Conversely, one can show that every tangent vector $X_p \in T_pM$ is the velocity vector of some curve at p, as follows.

Proposition 4.2.9. (Existence of a Curve with given initial vector). For any point p in a manifold M and any tangent vector $X_p \in T_pM$, there are $\epsilon > 0$ and a smooth curve $c : (-\epsilon, \epsilon) \to M$ such that $c(0) = p$ and $c'(0) = X_p$.

Proof : Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart centered at p, i.e , $\phi(p) = 0 \in \mathbb{R}^n$. Suppose $X_p = \sum a^i \, \partial/\partial x^i|_p$ at p. Let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then $x^i = r^i \circ \phi$. To find a curve c at p with $c'(0) = X_p$, start with a curve α in $\mathbb{R}^n$ with $\alpha(0) = 0$ and $\alpha'(0) = \sum a^i \, \partial/\partial r^i|_0$. We then map α to M via ϕ^{-1}. The simplest such α is :

$$\alpha(t) = (a^1 t, \ldots, a^n t), \quad t \in (-\epsilon, \epsilon),$$

where ϵ is sufficiently small that $\alpha(t)$ lies in $\phi(U)$. Define $c = \phi^{-1} \circ \alpha : (-\epsilon, \epsilon) \to M$. Then,

$$c(0) = \phi^{-1}(\alpha(0)) = \phi^{-1}(0) = p,$$

and we know,

$$c'(0) = (\phi^{-1})_* \alpha_* \left(\left. \frac{d}{dt} \right|_{t=0} \right) = (\phi^{-1})_* \left(\sum a^i \left. \frac{\partial}{\partial r^i} \right|_0 \right) = \sum a^i \left. \frac{\partial}{\partial x^i} \right|_p = X_p. \qquad \square$$

Proposition 4.2.10. Suppose X_p is a tangent vector at a point p of a manifold M and $f \in \mathbb{C}_p^\infty(M)$. If $c : (-\epsilon, \epsilon) \to M$ is a smooth curve starting at p with $c'(0) = X_p$, then

$$X_p f = \left. \frac{d}{dt} \right|_0 (f \circ c).$$

Proof : By the definition of $c'(0)$ and c_*,

$$X_p f = c'(0) f = c_* \left(\left. \frac{d}{dt} \right|_0 \right) f = \left. \frac{d}{dt} \right|_0 (f \circ c). \qquad \square$$

Proposition 4.2.11. (Computing the Differential Using Curves) Let $F : N \to M$ be a smooth map of manifolds, $p \in N$, and $X_p \in T_pN$. If c is a smooth curve starting at p in N with velocity X_p at p, then

$$F_{*,p}(X_p) \;=\; \left.\frac{d}{dt}\right|_0 (F \circ c)(t).$$

In other words, $F_{*,p}(X_p)$ is the velocity vector of the image curve $F \circ c$ at $F(p)$.

Proof : By hypothesis, $c'(0) = p$ and $c'(0) = X_p$. Then,

$$\begin{aligned}
F_{*,p}(X_p) &= F_{*,p}(c'(0)) \\
&= (F_{*,p} \circ c_{*,0})\left(\left.\frac{d}{dt}\right|_0\right) \\
&= (F \circ c)_{*,0}\left(\left.\frac{d}{dt}\right|_0\right) \quad \text{(by the chain rule)} \\
&= \left.\frac{d}{dt}\right|_0 (F \circ c)(t).
\end{aligned}$$

Chapter 5

Submanifolds

Many of the most familiar examples of manifolds arise naturally as subsets of other manifolds: for example, the n-sphere is a subset of $\mathbb{R}^{n+1}$ and the n-torus $\mathbb{T}^n = S^1 \times \ldots \times S^1$ is a subset of $\mathbb{C} \times \ldots \times \mathbb{C} = \mathbb{C}^n$. Goal of studying submanifolds is to find conditions under which a subset of a smooth manifold can be considered as a smooth manifold in its own right.

Definition 5.0.1. A subset S of a manifold N of dimension n is a regular **submanifold** of dimension k if for every $p \in S$, there is a coordinate neighborhood $(U, \phi) = (U, x^1, \ldots, x^n)$ of p in the maximal atlas of N such that $U \cap S$ is defined by the vanishing of $n-k$ of the coordinate functions. Such a chart (U, ϕ) in N, is called an adapted chart relative to S. On $U \cap S$, $\phi = (x^1, \ldots, x^k, 0, \ldots, 0)$.
Let , $\phi_S : U \cap S \to \mathbb{R}^k$ be the restriction of the first k components of ϕ to $U \cap S$ i.e $\phi_S := (x^1, \ldots, x^k)$. Hence, $(U \cap S, \phi_S)$ is a chart for S in the subspace topology.

Definition 5.0.2. $(n-k)$ is said to be the co dimension of S in N.

Proposition 5.0.1. Let S be a regular submanifold of N and $\mathcal{U} = \{(U, \phi)\}$ a collection of compatible adapted charts of N that cover S. Then, $\{(U \cap S, \phi_S)\}$ is an atlas for S. Therefore, a regular submanifold is itself a manifold. If N has dimension n and S is locally defined by the vanishing of $(n-k)$ coordinates then dim $S = k$.

Proof : Let $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be two adapted charts in the given collection.
For $p \in U \cap V \cap S$, $\phi(p) = (x^1, \ldots, x^k, 0, \ldots, 0)$, and $\psi(p) = (y^1, \ldots, y^k, 0, \ldots, 0)$ (by renumbering the co ordinates) , so $\phi_S(p) = (x^1, \ldots, x^k)$ and $\psi_S(p) = (y^1, \ldots, y^k)$.

$$\therefore \ (\psi_S \circ \phi_S^{-1})(x^1, \ldots, x^k) \ = \ (y^1, \ldots, y^k)$$

Since $y^1, \ldots, y^k$ are C^∞ functions of $x^1, \ldots, x^k$. Hence the transition functions $\phi_S \circ \phi_S^{-1}$ is C^∞. Similarly, $\phi_S \circ \psi_S^{-1}$ is C^∞.

Hence any two charts in $\{(U \cap S), \phi_S\}$ are C^∞ compatible.
Since $\{(U \cap S)\}_{U \in \mathcal{U}}$ covers S, the collection, $\{(U \cap S), \phi_S\}$ is a C^∞ atlas on S.

5.1 Immersion , Submersion, Embedding

Just as the derivative of a map between Euclidean Spaces is a linear map that approximates the given map at a point so the differential at a point serves the same purpose for a C^∞ map between manifolds.

Definition 5.1.1. A C^∞ map $F : N \to M$ is said to be an **immersion** at $p \in N$ if its differential $F_{*,p} : T_pN \to T_{F(p)}M$ is injective. F is **submersion** at $p \in N$ if $F_{*,p}$ is surjective.

F is said to be an immersion if it is immersion at every $p \in N$ and similarly F is said to be an submersion if it is submersion at every $p \in N$. Suppose dim $N = n$, dim $M = m$. Then, dim $T_pN = n$ and dim $T_{F(p)}M = m$. Injectivity of the map $F_{*,p}$ (immersion) implies $n \leq m$. Similarly, surjectivity of $F_{*,p}$ (submersion) implies $n \geq m$.

Example. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher dimensional $\mathbb{R}^m$:

$$i(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0).$$

The prototype of an submersion is the projection of $\mathbb{R}^n$ onto a lower dimensional $\mathbb{R}^m$:

$$\pi(x^1, \dots, x^m, x^{m+1}, \dots, x^n) = (x^1, \dots, x^m).$$

Note : If $U \subseteq M$, then the inclusion $i : U \to M$ is both an immersion and submersion so this shows submersion needs not be surjective.

Definition 5.1.2. One special kind of immersion is particularly important. A (smooth) embedding is an injective immersion $F : M \to N$ that is also a topological embedding, i.e., a homeomorphism onto its image $F(M) \subset N$ in the subspace topology. Since this is the primary kind of embedding we will be concerned with in this book, the term "embedding" will always mean smooth embedding unless otherwise specified.

Example : If $M_1, \dots, M_k$ are smooth manifolds and if $p_i \in M_i$ are arbitrarily chosen points, each of the maps $\tau_j : M_j \to M_1 \times \dots \times M_k$ given by

$$\tau_j(q) = (p_1, \dots, p_{j-1}, q, p_{j+1}, \dots, p_k)$$

is an embedding. In particular, the inclusion map $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ given by sending $(x_1, \dots, x_n)$ to $(x_1, \dots, x_n, 0, \dots, 0)$ is an embedding.

5.2 Rank, Regular and Critical Points :

Consider a smooth map $F : N \to M$ of manifolds. It's rank at a point p in N, denoted by rk $F(p)$ is defined as the rank of the differential,

$$F_{*,p} : T_pN \to T_{F(p)}M$$

Since $F_{*,p}$ is represented by the Jacobian matrix,

$$rkF(p) = rk\left[\frac{\partial F^i}{\partial x^j}(p)\right]$$

Since differential of a map is independent of coordinate charts, so is rank of a Jacobian matrix.

Definition 5.2.1. A point p in N is a **critical point** of F if the differential $F_{*,p} : T_pN \to T_{F(p)}M$ fails to be surjective. A point in M is **critical value** if it is image of a critical point. It is a **regular point** of F if the differential $F_{*,p}$ is surjective. In other words, p is a regular point of the map F if and only if F is a submersion at p. A point in M is a **regular value** if it is not a critical value.

Remarks : Regular value doesn't need to be image of regular point or image of F at all. A point c in M is a critical value iff some point in F^{-1} is a critical point. A point c in image of F is regular iff every $F^{-1}(\{c\})$ is a regular point.

Proposition 5.2.1. For : $M \to \mathbb{R}$, a point p in M is a critical point if relative to some chart $(U, x^1, \ldots, x^n)$ containing p, all the derivatives satisfy :

$$\frac{\partial f}{\partial x^j}(p) = 0 \ , \qquad j = 1(1)n$$

Proof : $f_{*,p} : T_pM \to T_{F(p)}\mathbb{R}$ represents by the matrix :

$$\left[\frac{\partial f}{\partial x^1}(p), \ldots, \frac{\partial f}{\partial x^n}(p)\right]$$

Since image of $f_{*,p}$ is linear subspace of $\mathbb{R}$ hence either 0 or 1 dimensional, i.e $f_{*,p}$ is either zero map or surjective map.
$\therefore$ $f_{*,p}$ fails to be surjective when $\dfrac{\partial f}{\partial x^i}(p) = 0 \ \ \forall j$ and then p is a critical point.

5.3 Level Sets of a Function :

Definition 5.3.1. A **level set** of a map $F : N \to M$ is a subset

$$F^{-1}(\{c\}) \ = \ \{p \in N | F(p) = c\}$$

for some $c \in M$. The value $c \in M$ is called *level* of the level set $F^{-1}(\{c\})$. $F^{-1}(0)$ is called *zero set* of F.

Example. Many submanifolds are most naturally defined as the set of points where some smooth map takes on a fixed value, called a level set of the map. For example, the n-sphere $S^n \subset \mathbb{R}^{n+1}$ is defined as the level set $f^{-1}(1)$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is the function $f(x) = |x|^2$.

Remarks. The inverse image $F^{-1}(c)$ of a regular value c is called a *regular level set.* If the zero set $F^{-1}(0)$ is a regular level set of $F : N \to \mathbb{R}^m$, it's called a *regular zero set.*

If a regular level set $F^{-1}(c)$ is non empty , say $p \in F^{-1}(c)$, then the map $F : N \to M$ is a submersion at p. Hence, dim $N \geq$ dim M.

Example. (The 2-sphere in $\mathbb{R}^3$) The unit 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ is the level set $g^{-1}(1)$ of level 1 of the function $g(x, y, z) = x^2 + y^2 + z^2$. We will use the inverse function theorem to find adapted charts of $\mathbb{R}^3$ that cover S^2 . As the proof will show, the process is easier for a zero set, mainly because a regular submanifold is defined locally as the zero set of coordinate functions. To express S^2 as a zero set, we rewrite its defining equation as $f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Then $S^2 = f^{-1}(0)$.

This is an important example because one can generalize its proof almost verbatim to prove that if the zero set of a function $f : N \to \mathbb{R}$ is a regular level set, then it is a regular submanifold of N. The idea is that in a coordinate neighborhood $(U, x^1, \ldots, x^n)$ if a partial derivative $\partial f / \partial x^i (p)$ is nonzero, then we can replace the coordinate x^i by f . First we show that any regular level set $g^{-1}(c)$ of a C^∞ real function g on a manifold can be expressed as a regular zero set.

Lemma 5.3.1. Let $g : N \to \mathbb{R}$ be a C^∞ function. A regular level set $g^{-1}(c)$ of level c of the function g is the regular zero set $f^{-1}(0)$ of the function $f = g - c$.

Proof : For any $p \in N$, $g(p) = c \iff f(p) = g(p) - c = 0$
Hence , $g^{-1}(c) = f^{-1}(0)$. Call this set S because $f_{*,p} = g_{*,p}$ at every point $p \in N$, the functions f and g will have exact same critical points. Since g has no critical points in S, neither does f.

Next it can be proven that a nonempty regular set $g^{-1}(c)$ corresponding to a C^∞ function g is a regular submanifold of co-dimension 1. The next step is to extend to a regular level set of a map between smooth manifolds. This very useful theorem does not seem to have an agreed-upon name in the literature. It is known variously as the implicit function theorem, the pre-image theorem, and the regular level set theorem, among other nomenclatures. But we will not discuss all of these further in this project.

Chapter 6

Conclusion and scope of further study

Since a manifold in general does not have standard coordinates, only coordinate-independent concepts makes sense on a manifold. For example, it turns out that on a manifold of dimension n, it is not possible to integrate functions, because the integral of a function depends on a set of coordinates. The objects that can be integrated are differential forms. It is only because the existence of global coordinates permits an identification of functions with differential n-forms on $\mathbb{R}^n$ that integration of functions becomes possible on $\mathbb{R}^n$.

6.1 Differential Form on Manifold

Differential forms are generalizations of real-valued functions on a manifold. Instead of assigning to each point of the manifold a number, a differential k-form assigns to each point a k-covector on its tangent space. For $k = 0$ and 1, differential k-forms are functions and covector fields respectively. In contrast to vector fields, which are also intrinsic to a manifold, differential forms have a far richer algebraic structure.

Because integration of functions on a Euclidean space depends on a choice of coordinates and is not invariant under a change of coordinates, it is not possible to integrate functions on a manifold. The highest possible degree of a differential form is the dimension of the manifold. Among differential forms, those of top degree turn out to transform correctly under a change of coordinates and are precisely the objects that can be integrated. The theory of integration on a manifold would not be possible without differential forms.

6.1.1 Differential 1 Form

Let M be a smooth manifold and p is a point in M. The **cotangent space** of M at p denoted by $T_p^*(M)$ or T_p^*M is defined to be the dual space of the tangent space T_pM :

$$T_p^*M = (T_pM)^V = \mathrm{Hom}(T_pM, \mathbb{R}).$$

An element of the cotangent space T_p^*M is called a **covector** at p. Thus a covector ω_p at p is linear function

$$\omega_p : T_pM \to \mathbb{R}$$

A **covector field**, a **differential 1 form** is a function ω that assigns to each point p in M a covector ω_p at p. In this sense it is dual to a vector field on M, which assigns to each point in M a tangent vector at p.

Definition 6.1.1. If f is a C^∞ real valued function on a manifold M, its **differential** is defined to be the 1 form df on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(df)_p(X_p) \;=\; X_pf$$

Proposition 6.1.1. If $f : M \to \mathbb{R}$ is a C^∞ function, then for $p \in M$ and $X_p \in T_pM$,

$$f_*(X_p) = (df)_p(X_p)\frac{d}{dt}\bigg|_f (p)$$

Proof. Since $f_*(X_p) \in T_{f(p)}\mathbb{R}$, there is a real number a such that,

$$f_*(X_p) = a\frac{d}{dt}\bigg|_{f(p)}$$

To evaluate a, apply both sides of the above equation to x :

$$a = f_*(X_p)(t) = X_p(t \circ f) = X_pf = (df)_pX_p$$

This proposition shows that under the canonical identification of the tangent space $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$ via $a{}^{d}\!/\!{}_{dt}|_{f(p)} \;\leftrightarrow a,\;\; f_*$ is the same as df.

We can generalize the construction of 1-forms on a manifold to k-forms. After defining k-forms on a manifold, we show that locally they look no different from k-forms on $\mathbb{R}^n$.

6.2 Partition of Unity

A partition of unity on a manifold is a collection of non negative functions that sum to 1. Usually one demands in addition that the partition of unity be subordinate to an open cover $\{U_\alpha\}_{\alpha\in A}$. The existence of a C^∞ partition of unity is one of the most important technical tools in the theory of C^∞ manifolds. A partition of unity is used in two ways: (1) to decompose a global object on a manifold into a locally finite sum of local objects on the open sets U_α of an open cover, and (2) to patch together local objects on the open sets U_α into a global object on the manifold. Thus, a partition of unity serves as a bridge between global and local analysis on a manifold. This is useful because while there are always local co-ordinates on a manifold, there may be no global coordinates. In subsequent sections we will see examples of both uses of a C^∞ partition of unity.

If $\{U_i\}_{i\in I}$ is a finite open cover of M, a C^∞ *partition of unity subordinate to* $\{U_i\}_{i\in I}$ is a collection of non negative C^∞ functions $\{\rho_i : M \to \mathbb{R}\}_{i\in I}$ such that supp $\rho_i \subset U_i$ and $\sum \rho_i = 1$. When I is an infinite set, for the sum to make sense, we will impose a *local finiteness* condition. A collection $\{A_\alpha\}$ of subsets of a topological space S is said to be *locally finite* if every point q in S has a neighborhood that meets only finitely many of the sets A_α . In particular, every q in S is contained in only finitely many of the A_α 's.

Definition 6.2.1. A C^∞ partition of unity on a manifold is a collection of non negative C^∞ functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha\in A}$ such that

(i) the collection of supports, $\{\text{supp } \rho_\alpha\}_\alpha \in A$, is locally finite,

(ii) $\sum \ \rho_\alpha = 1$.

Given an open cover $\{U_\alpha\}_{\alpha\in A}$ of M, we say that a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ is subordinate to the open cover $\{U_\alpha\}$ if supp $\rho_\alpha \subset U_\alpha$ for every $\alpha \in A$.

We can also prove he existence of a C^∞ partition of unity on a manifold but we will not discuss it here.

Lastly let us look forward to what more we can achieve by extending the concepts we discussed in this project. The modern definition of a differential form as a section of an exterior power of the cotangent bundle (not discussed in this project) appeared in the late forties, after the theory of fiber bundles (not discussed) came into being. Interestingly, we can use differential and integral calculus on manifolds to study the topology of manifolds. We obtain a more refined invariant called the de Rham cohomology (not discussed) of the manifold. Due to the existence of the wedge product, a grading, and the exterior derivative, the set of smooth forms on a manifold is both a graded algebra and a differential complex. Such an algebraic structure is called a differential graded algebra. Moreover, the differential complex of smooth forms on a manifold can be pulled back under a smooth map, making the complex into a contravariant functor called the de Rham complex of the manifold.

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