An Introduction to Manifolds

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Contents

Goal

1. This project is about the extension of calculus from curves and surfaces to higher dimensions. The higher-dimensional analogues of smooth curves and surfaces are called manifolds.
2. We generalize the notion of directional derivative on $\mathbb{R}^n$ by introducing equivalence relation on C^∞ functions in the neighborhood of a point p and call the linear map of directional derivative as derivation at p.
3. We discuss the notion of a smooth manifold and smooth maps between two manifolds. Using coordinate charts, one can transfer the notion of smooth maps from Euclidean spaces to manifolds.
4. A smooth map of manifolds induces a linear map, called its differential, of tangent spaces at corresponding points. In local coordinates, the differential is represented by the Jacobian matrix of partial derivatives of the map.
5. We introduce the concept of submanifolds, immersion, submersion maps on a manifold and rank, critical, regular points and level set.

What is a Manifold?

Preliminary concepts required to define a Manifold

A topological space is **second countable** if it has a countable basis. A **neighborhood** of a point p in a topological space M is any open set containing p. An **open cover** of M is a collection $\{U_\alpha\}_{\alpha \in A}$ of open sets in M whose union $\bigcup_{\alpha \in A} U_\alpha$ is M. A topological space M is **locally Euclidean** of dimension n if every point p in M has a neighborhood U such that there is a **homeomorphism** ϕ from U onto an open subset of $\mathbb{R}^n$.

Definition

A topological **manifold** is a Hausdorff, second countable, locally Euclidean space. Manifold is of dimension n if it is locally Euclidean of dim n.

- U is called *coordinate neighborhood* .
- ϕ is called *coordinate map* on U.
- $(U \ , \ \phi : U \to \mathbb{R}^n)$ is called a *chart*.


Figure: Sphere


Figure: Torus


Figure: Double Torus

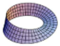


Figure: Mobius Strip


Figure: Klein Bottle

Figure: Different types of Manifolds

Transition Functions

Two charts $(U\ ,\ \phi : U \to \mathbb{R}^n)$ and $(V\ ,\ \psi : U \to \mathbb{R}^n)$ are C^∞ **compatible** if these two maps are C^∞ maps, they are called **transition functions**.

$$\begin{aligned} \phi \circ \psi^{-1} &: \psi(U \cap V) \longrightarrow \phi(U \cap V) \\ \psi \circ \phi^{-1} &: \phi(U \cap V) \longrightarrow \psi(U \cap V) \end{aligned}$$

- $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are called **transition functions** between the charts.
- If $U \cap V$ is empty, they are automatically C^∞ compatible.

A **'C^∞-Atlas'** on a locally Euclidean space M is a collection $U = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞ compatible charts that cover M i.e, $M = \bigcup_\alpha U_\alpha$. An atlas $\mathcal{A}$ on a locally Euclidean space is said to be **maximal atlas** if it is not contained in a larger atlas i.e, if $\mathcal{B}$ is another atlas containing $\mathcal{A}$ then $\mathcal{B} = \mathcal{A}$.

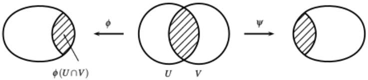


Figure: The Transition function $\psi \circ \phi^{-1}$ is defined on $\phi(U \cap V)$

Smooth Manifold

Smooth Manifold

A manifold is **smooth manifold** or C^∞ manifold if the transition maps are smooth. It is a topological manifold together with a maximal atlas (also called differentiable structure on M). To show, that a topological space M is a C^∞ manifold, it suffices to check that,

- M is Hausdorff and second countable,
- M has a C^∞ atlas.

Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be a C^∞ or **smooth map** at a point $p \in M$ if there is a chart (U, ϕ) about $p \in M$ such that , $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^n$ is C^∞ at $\phi(p)$. f is said to be C^∞ on M if f is C^∞ at every point of M.

In the context of manifolds, we denote the standard co-ordinates on $\mathbb{R}^n$ by $r^1, \dots, r^n$. If $(U, \phi : U \to \mathbb{R}^n)$ is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the i-th component of ϕ and write $\phi = (x^1, \dots, x^n)$ and $(U, \phi) = (U, x^1, \dots, x^n)$. Thus, for $p \in U$, $(x^1(p), \dots, x^n(p))$ is a point in $\mathbb{R}^n$. The functions $x^1, \dots, x^n$ are called local coordinates on U.

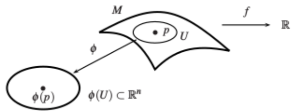


Figure: Checking that a function f is C^∞ at p by pulling back to $\mathbb{R}^n$.

Smoothness of function independent of Chart

The definition of smoothness of a function f at a point is independent of the chart (U, ϕ). For if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, ψ) be any other chart about $p \in M$, then on $\psi(V \cap U)$, the function

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$$

is also C^∞ at $\phi(p)$.

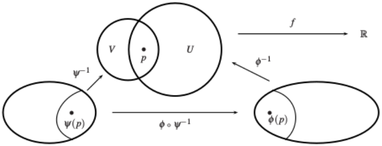


Figure: Checking that a function f is C^∞ at p via two charts.

Corollary

If M be a manifold of dimension n and $f : M \to \mathbb{R}$ is a *real valued* C^∞ function on M, then for every chart (V, ψ) on M, $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞ on $\psi(V)$.

Smooth Map between two Manifolds

Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ at a point p in N if there are charts (V, ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$. The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.

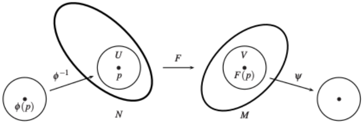


Figure: Checking that a map $F : N \to M$ is C^∞ at p.

We assume $F : N \to M$ continuous to ensure that $F^{-1}(V)$ is an open set in N. Thus, C^∞ maps between manifolds are by definition continuous.

Product Manifold

We have (finite) products in the category of manifolds. Say, M and N are m and n-dim C^∞ manifolds, respectively, then the product space $M \times N$ into an $m+n$-dim C^∞ manifold. If we have a coordinate patch (U, ϕ_U) on M and another (V, ϕ_V) on N, then we surely have $U \times V \subseteq M \times N$ as an open subset of the product space. We just define

$$\phi_{U\times V} = \phi_U \times \phi_V : U \times V \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$$

If U' and V' are another pair of coordinate patches we can set up the transition function

$$\phi_{U'\times V'} \circ \phi^{-1}_{U\times V} = (\phi_{U'} \times \phi_{V'}) \circ (\phi_U \times \phi_V) = (\phi_{U'} \circ \phi_U^{-1}) \times (\phi_{V'} \circ \phi_V^{-1})$$

Each of these factors is smooth on M and N. Since smoothness is determined component-wise, the product mapping is smooth as well. So we have an atlas making $M \times N$ a C^∞ manifold. The collection $\{(U_i x V_j, \phi_{U_i} \times \phi_{V_j})\}$ of charts is an atlas on $M \times N$.

Figure: the infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are product manifolds

Jacobian Matrix

Partial Derivative

On a manifold M of dimension n , let (U, ϕ) be a chart and f a C^∞ function. As a function into $\mathbb{R}^n$, ϕ has n components $x^1, \dots, x^n$. If $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then $x^i = r^i \circ \phi$.

For, $p \in U$, we define the **partial derivative** $\dfrac{\partial f}{\partial x^i}$ of f with respect to x^i at p to be :

$$\frac{\partial}{\partial x^i}\bigg|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)) := \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} (f \circ \phi^{-1})$$

Jacobian Matrix of a function between two manifolds

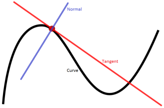
Let $F : N \to M$ be a smooth map , and let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by :

$$F^i := y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

is the i th component of F in the chart (V, ψ). Then the matrix $\left[\dfrac{\partial F^i}{\partial x^j}\right]$ is called the **Jacobian matrix** of F relative to the charts (U, ϕ) and (V, ψ). The transition map $\phi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^{\ltimes}$. It's Jacobian matrix $J(\psi \circ \phi^{-1})$ at $\phi(p)$ is the matrix $\left[\dfrac{\partial y^i}{\partial x^j}\right]$ of partial derivatives at p.

Concept of Tangent Vector to a Surface

Our Ordinary Concept of Tangent in an Example :



Let the points of $\mathbb{R}^3$ be denoted by (x, y, z) and consider the sphere M whose equation is $x^2 + y^2 + (z-1)^2 = 1$. The point $p = (0, 0, 0)$ lies on M, and for any real number a and b both not 0, the line $\{(au, bu, 0) : -\infty < u < \infty\}$ is a tangent to M at p. Say, $a = 1$, $b = 2$, then the line $y = 2x$ is tangent line restricted to $x - y$ plane and $(1, 2, 0)$ is tangent vector of M at p.

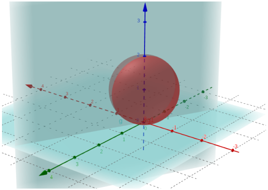


Figure: Intersection of planes $y = 2x$ and $z = 0$ is one such tangent vector $\{(u, 2u, 0)\}$ at $(0, 0, 0)$

A basic principle in manifold theory is the linearization principle, according to which a manifold can be approximated near a point by its tangent space at the point. However, this elementary analytic-geometric notion does not extend very well to arbitrary differentiable manifolds. Instead, tangent vectors will be defined either as directional derivatives or point-derivation or a velocity vector at the point.

Tangent Vector as Directional Derivative

In calculus we visualize the tangent space at p in $\mathbb{R}^n$ denoted by $T_p(\mathbb{R}^n)$ as the vector space of all arrows emanating from p. The line through a point $p = (p^1, \ldots, p^n)$ with direction $v = < v^1, \ldots, v^n >$ in $\mathbb{R}^n$ has parametrization :

$$c(t) = (p^1 + tv^1, \ldots, p^n + tv^n).$$

If f is C^∞ in a neighborhood of p in $\mathbb{R}^n$ and v is a tangent vector at p, the **directional derivative** of f in the direction v at p is defined to be

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \frac{d}{dt}\bigg|_{t=0} f(c(t))$$

By the chain rule,

$$D_v f = \sum_{i=1}^{n} \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p)$$

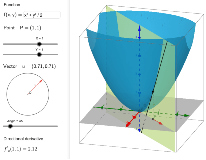


Figure: Example : Directional Derivative

The association $v \mapsto D_v$ of the directional derivative D_v to a tangent vector v offers a way to characterize tangent vectors as certain operators on functions.

Defining Tangent Space via Tangent Curve

Let M is a C^k manifold and $x \in M$. Pick a coordinate chart $\phi : U \to \mathbb{R}^n$, where $x \in U \subseteq M$ is open. Let, $\gamma_1, \gamma_2 : (-1, 1) \to M$ with $\gamma_1(0) = x = \gamma_2(0)$ be two curves initialized at x, such that $\phi \circ \gamma_1, \phi \circ \gamma_2 : (-1, 1) \to \mathbb{R}^n$ are differentiable. Then γ_1 and γ_2 are said to be equivalent at 0 if and only if the derivatives of $\phi \circ \gamma_1$ and $\phi \circ \gamma_2$ at 0 coincide. This defines an equivalence relation on the set of all differentiable curves initialized at x, and equivalence classes of such curves are known as tangent vectors of M at x. The equivalence class of any such curve γ is denoted by $\gamma'(0)$.

The tangent space of M at x, denoted by $T_x M$, is then defined as the set of all tangent vectors at x;independent of the choice of coordinate chart $\phi : U \to \mathbb{R}^n$.

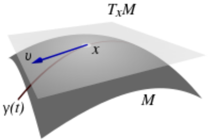


Figure: The tangent space $T_x M$ and a tangent vector $v \in T_x M$, along a curve traveling through $x \in M$.

To define vector-space operations on $T_x M$, we use a chart $\phi : U \to \mathbb{R}^n$ and define a map $d\phi_x : T_x M \to \mathbb{R}^n$ by:

$$d\phi_x(\gamma'(0)) \overset{\text{df}}{=} \frac{d}{dt}[(\phi \circ \gamma)(t)]\Big|_{t=0}, \quad \text{where } \gamma \in \gamma'(0).$$

This construction does not depend on the particular chart $\phi : U \to \mathbb{R}^n$ and the curve γ being used. The map $d\phi_x$ turns out to be bijective and may be used to transfer the vector-space operations on $\mathbb{R}^n$ over to $T_x M$, thus turning the latter set into an n-dimensional real vector space.

Defining Tangent Space via Point-Derivation

Germ

Consider the set of all pairs (f, U), where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. $(f, U) \equiv (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. The equivalence class of (f, U) is called the **Germ** of f at p. The set of all germs of C^∞ functions on $\mathbb{R}^n$ at p is $C_p^\infty(\mathbb{R}^n)$.

Point Derivation

For each tangent vector v at a point p in $\mathbb{R}^n$, the directional derivative at p gives a map of real vector spaces $D_v : C_p^\infty \to \mathbb{R}$. D_v is $\mathbb{R}$-linear and satisfies the Leibniz rule : $D_v(fg) = (D_v f)g(p) + f(p)D_v g$.
Any linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule is called a **derivation** at p or a **point-derivation** of C_p^∞.
Since, directional derivatives at p are all derivations at p, so there is a map,

$$v \mapsto D_v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p$$

Tangent Space as point-derivation

A **tangent vector** at a point p in a manifold M is a *derivation at* p. The tangent vectors at p form a vector space $T_p(M)$, called the tangent space of M at p. If U is an open set containing p in M, then the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$. Hence, $T_pU = T_pM$.

Differential of a Map

Suppose $F:\mathbb{R}^n \to \mathbb{R}^m$ is smooth and p is a point in $\mathbb{R}^n$. Let $x^1,\dots,x^n$ be the coordinates on $\mathbb{R}^n$ and $y^1,\dots,y^m$ be the coordinates on $\mathbb{R}^m$. Then the tangent vectors $\left.\frac{\partial}{\partial x^1}\right|_p,\dots,\left.\frac{\partial}{\partial x^n}\right|_p$ form a basis for the tangent space $T_p(\mathbb{R}^n)$ and $\left.\frac{\partial}{\partial y^1}\right|_{F(p)},\dots,\left.\frac{\partial}{\partial y^m}\right|_{F(p)}$ form a basis for the tangent space $T_F(p)(\mathbb{R}^m)$.

The linear map $F_*: T_p(\mathbb{R}^n) \to T_F(p)(\mathbb{R}^m)$ is described by the matrix $[a^i_j]$ relative to these two bases :

$$F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)} \quad , \quad a^k_j \in \mathbb{R}$$

Let $F: N \to M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map F induces a linear map of tangent spaces called its differential at p, $F_*: T_pN \to T_{F(p)}M$. If $X_p \in T_pN$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined as :

$$(F_*(X_p))(f) = X_p(f \circ F) \in \mathbb{R} \qquad \text{for } f \in C^\infty_{F(p)}(M)$$

f is a germ at $F(p)$, represented by a C^∞ function in a neighborhood of $F(p)$.

The Chain Rule : If $F: N \to M$ and $G: M \to P$ are smooth map of manifolds and $p \in N$, then,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}.$$

Existence of a Curve with given initial vector

A **smooth curve** in a manifold M is by definition a smooth map $c : (a, b) \to M$ from some open interval (a, b) into M. Usually we assume $0 \in (a, b)$ and say that c is a curve starting at p if $c(0) = p$. The *velocity vector* $c'(t_0)$ of the curve c, which is the velocity of c at the point $c(t_0)$ at time $t_0 \in (a, b)$ is defined to be

$$c'(t_0) := c_* \left(\left. \frac{d}{dt} \right|_{t_0} \right) \in T_{c(t_0)} M.$$

Velocity of a Curve in Local Coordinates : Let $c : (a, b) \to M$ be a smooth curve, and let $(U, x^1, \dots, x^n)$ be a coordinate chart about $c(t)$. Set $c^i = x^i \circ c$ for the i th component of a c in the chart. Then $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^{n} c^i(t) \left. \frac{\partial}{\partial x^i} \right|_{c(t)}.$$

Every smooth curve c at p in a manifold M gives rise to a tangent vector $c'(0)$ in T_pM. Conversely, one can show that every tangent vector $X_p \in T_pM$ is the velocity vector of some curve at p.

Existence of a Curve with given initial vector : For any point p in a manifold M and any tangent vector $X_p \in T_pM$, there are $\epsilon > 0$ and a smooth curve $c : (-\epsilon, \epsilon) \to M$ such that $c(0) = p$ and $c'(0) = X_p$.

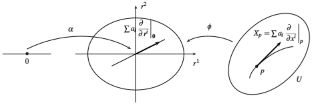


Figure: Existence of a curve through a point with a given initial vector.

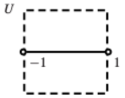
How to define a Sub-Manifold?

Submanifold

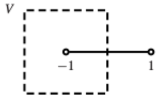
A subset S of a manifold N of dimension n is a **regular submanifold** of dimension k if for every $p \in S$, there is a coordinate neighborhood $(U, \phi) = (U, x^1, \dots, x^n)$ of p in the maximal atlas of N such that $U \cap S$ is defined by the vanishing of $n - k$ of the coordinate functions. Such a chart (U, ϕ) in N, is called an **adapted chart** relative to S. On $U \cap S$, $\phi = (x^1, \dots, x^k, 0, \dots, 0)$.

Let , $\phi_S : U \cap S \to \mathbb{R}^k$ be the restriction of the first k components of ϕ to $U \cap S$ i.e $\phi_S := (x^1, \dots, x^k)$. Hence, $(U \cap S, \phi_S)$ is a chart for S in the subspace topology. Then, $(n - k)$ is said to be the **co-dimension** of S in N.

Example : The interval $S :=]-1, 1[$ on the x-axis is a regular submanifold of the xy-plane. As an adapted chart, we can take the open square $U =]-1, 1[\times]-1, 1[$ with coordinates x, y. Then $U \cap S$ is precisely the zero set of y on U. But, $V =]-2, 0[\times]-1, 1[$, then (V, x, y) is not an adapted chart relative to S, since $V \cap S$ is the open interval $]-1, 0[$ on the x-axis, while the zero set of y on V is the open interval $]-2, 0[$ on the x-axis.



U is an adapted chart.



V is not an adapted chart.

Few Important Concepts

Immersion, Submersion

A C^∞ map $F : N \to M$ is said to be an **immersion** at $p \in N$ if its differential $F_{*,p} : T_pN \to T_{F(p)}M$ is injective. F is **submersion** at $p \in N$ if $F_{*,p}$ is surjective. F is said to be an immersion if it is immersion at every $p \in N$. Suppose dim $N = n$, dim $M = m$. Then, dim $T_pN = n$ and dim $T_{F(p)}M = m$. Injectivity of the map $F_{*,p}$ (immersion) implies $n \leq m$. Similarly, surjectivity of $F_{*,p}$ (submersion) implies $n \geq m$.

Example. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher dimensional $\mathbb{R}^m$:
$i(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$.
The prototype of an submersion is the projection of $\mathbb{R}^n$ onto a lower dimensional $\mathbb{R}^m$:
$\pi(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = (x^1, \ldots, x^m)$.

Rank

Consider a smooth map $F : N \to M$ of manifolds. It's rank at a point p in N, denoted by rk $F(p)$ is defined as the rank of the differential, $F_{*,p} : T_pN \to T_{F(p)}M$. Since $F_{*,p}$ is represented by the Jacobian matrix,

$$rkF(p) = rk\left[\frac{\partial F^i}{\partial x^j}(p)\right]$$

Since differential of a map is independent of coordinate charts, so is rank of a Jacobian matrix.

Few Important Concepts

Critical Point, Regular Point

A point p in N is a **critical point** of F if the differential $F_{*,p} : T_pN \to T_{F(p)}M$ fails to be surjective. A point in M is **critical value** if it is image of a critical point. It is a **regular point** of F if the differential $F_{*,p}$ is surjective. In other words, p is a regular point of the map F if and only if F is a submersion at p. A point in M is a **regular value** if it is not a critical value.

Level Set

A **level set** of a map $F : N \to M$ is a subset $F^{-1}(\{c\}) = \{p \in N | F(p) = c\}$ for some $c \in M$. The value $c \in M$ is called *level* of the level set $F^{-1}(\{c\})$. $F^{-1}(0)$ is called *zero set* of F.

The inverse image $F^{-1}(c)$ of a regular value c is called a *regular level set.* If the zero set $F^{-1}(0)$ is a regular level set of $F : N \to \mathbb{R}^m$, it's called a *regular zero set.*

Example. (The 2-sphere in $\mathbb{R}^3$) The unit 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ is the level set $g^{-1}(1)$ of level 1 of the function $g(x, y, z) = x^2 + y^2 + z^2$. We will use the inverse function theorem to find adapted charts of $\mathbb{R}^3$ that cover S^2 . As the proof will show, the process is easier for a zero set, mainly because a regular submanifold is defined locally as the zero set of coordinate functions. To express S^2 as a zero set, we rewrite its defining equation as $f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Then $S^2 = f^{-1}(0)$.

Further Scope of Study

- Since a manifold in general does not have standard coordinates, only coordinate-independent concepts makes sense on a manifold. For example, on a manifold of dimension n, it is not possible to integrate functions. The objects that can be integrated are differential forms which are generalizations of real-valued functions on a manifold. Instead of assigning to each point of the manifold a number, a differential k-form assigns to each point a k-covector on its tangent space. For $k = 0$ and 1, differential k-forms are functions and covector fields respectively.
- Many of the problems in mathematics share common features. This has given rise to the theory of categories and functors, which tries to clarify the structural similarities among different areas of mathematics. A category is essentially a collection of objects and arrows between objects. Smooth manifolds and smooth maps form a category, and so do vector spaces and linear maps.
- A Lie group is a manifold that is also a group such that the group operations are smooth. Lie's original motivation was to study the group of transformations of a space as a continuous analogue of the group of permutations of a finite set. Indeed, a diffeomorphism of a manifold M can be viewed as a permutation of the points of M.

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THANK YOU