Partial Fields

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Outline

Why Study Partial Fields?

Partial Fields were introduced by Semple and Whittle (1996) to study generalizations of totally unimmodular matrices and regular matroids in a systematic way. It is shown that if A is a matrix over a partial field that has the property that all of its square submatrices have defined determinants, then a well-defined matroid can be associated with A.

Definitions I

Partial Functions

A partial function on a set S is a function whose domain is a subset of S.

Partial Binary Operations

A partial binary operation on S is a function $+ : A \to S$ whose domain is a subset A of $S \times S$. If $(a, b) \in A$, then $a + b$ is defined, otherwise it is not defined.

Association

Let S be a set with a commutative partial binary operation $+$. Say S' is a finite multiset of elements of S. An association of the multiset S' is a way of unambiguously defining sums to obtain an expression that is a version of the sum of the elements of S'. To say that $a_1 + a_2 + \ldots + a_n$ is defined it means some association of $\{a_1, a_2, \ldots, a_n\}$ has all the sums defined.

Ring Definition of a Partial Field

A partial field is a pair (R, G), where R is a commutative ring, and G is a subgroup of R^* such that $-1 \in G$.

If $\mathbb{P} = (R, G, +, \cdot, 0, 1)$ is a partial field, and $p \in R$, then we say that p is an element of $\mathbb{P}$ if $p = 0$ or $p \in G$. We define $\mathbb{P}^* := G = P - \{0\}$. Multiplication "$\cdot$" is a binary function but addition "$+$" is a partial binary function. Clearly, if $p, q \in \mathbb{P}$ then also $p \cdot q \in \mathbb{P}$, but $p + q$ need not be an element of $\mathbb{P}$. A partial field is trivial if $1 = 0$.

More Concepts I

Determinant

Let A be an $n \times n$ square matrix with entries in a partial field $\mathbb{P}$. Just as with fields, we define the determinant to be a signed sum of products determined by permutations. Let p be an element of S_n, the group of permutations of $\{1, 2, \ldots, n\}$. Then $\varepsilon(p)$ denotes the sign of p. Formally, the determinant of A is defined by

$$\det(A) = \sum_{p \in S_n} \varepsilon(p) a_{1p(1)} a_{2p(2)} \cdots a_{np(n)},$$

if this sum is defined.

More Concepts II

$\mathbb{P}$-matrix

An $m \times n$ matrix A over a partial field $\mathbb{P}$ is a $\mathbb{P}$-matrix or weak $\mathbb{P}$-matrix if $\det(A')$ is defined for every square submatrix A' of A. Say A is a $\mathbb{P}$-matrix; then a non-empty set of columns $\{c_{i_1}, c_{i_2}, \ldots, c_{i_k}\}$ of A is independent if $k \leq m$, and at least one of the $k \times k$ submatrices of A with columns indexed by $\{i_1, i_2, \ldots, i_k\}$ has a non-zero determinant. Also, an empty set of columns is independent. The independent sets of vectors of a $\mathbb{P}$-matrix are the independent sets of a matroid. We consider matrices whose columns are labelled by the elements of a set E. A subset of E is independent if the set of columns it labels is independent.

More Concepts III

Nondegenerate Matrix

An $r \times E$ weak $\mathbb{P}$-matrix A is nondegenerate if there exists an $X \subseteq E$ with $|X| = r$ and $\det(A[r, X]) \neq 0$. Note that A is always degenerate if $\mathbb{P}$ is trivial.

Basis

Define:

$$\mathcal{B} := \{X \subseteq E \mid |X| = r, \det(A[r, X]) \neq 0\}.$$

Then $\mathcal{B}$ is the set of bases of a matroid in the partial field $\mathbb{P}$.

More Concepts

Pivot

Let x_{st} be a non-zero entry of a matrix A. Recall that a pivot on x_{st} is obtained by multiplying row s by $1/x_{st}$ and, for i in $\{1, 2, \ldots, s-1, s+1, \ldots, m\}$, replacing x_{ij} by $x_{st}^{-1} \begin{vmatrix} x_{st} x_{sj} \\ x_{it} x_{ij} \end{vmatrix}$.

P-Representable Matrix

If A is a P-matrix for some partial field P, whose columns are labelled by a set S. Then the independent subsets of S are the independent sets in a matroid on S. This matroid is denoted by $M[A]$. A matroid M is representable over P or is P-representable if it is equal to $M[A]$ for some P-matrix A, and A is called a representation of M.

Standard Form of Matroid Representation

A matroid representation of the form $[I|A]$ where I is the identity matrix is said to be in standard form.

Properties I

PROPOSITION 3.3. Let A be a P-matrix. If the matrix B is obtained from A by one of the following operations, then B is a P-matrix:

(i) interchanging a pair of rows or columns;

(ii) replacing a row or column by a non-zero scalar multiple of that row or column;

(iii) performing a pivot on a non-zero entry of A.

LEMMA 3.4. Let A be an $n \times (n+1)$ P-matrix, where $n \geq 2$, and assume that each row of A has a non-zero entry. Let B be an $n \times n$ submatrix of A. If all other $n \times n$ submatrices of A have zero determinant, then $\det B \neq 0$.

PROPOSITION 3.5. The *independent sets of a P-matrix* are preserved under the operations of interchanging a pair of rows or columns, multiplying a column or a row by a non-zero scalar, and performing a pivot on a non-zero entry of the matrix.

Properties II

PROPOSITION 4.1. If the matroid M is **representable over P** then M can be represented by a P-matrix of the form $[I|A]$, where I is an identity matrix. A representation of the form $[I|A]$ is said to be the Standard Form.

PROPOSITION 4.2. Let M and N be matroids representable over P.

(i) M^* is representable over P.

(ii) All minors of M are representable over P.

(iii) The direct sum of M and N is representable over P.

PROPOSITION 4.3. The matroid M is a (G, F)-matroid if and only if it is representable over (G, F).

Examples of Partial Fields

1. The partial field $GF(3) = (\{-1, 1\}, GF(3))$ and Reg $= (\{-1, 1\}, \mathbb{Q})$ aka regular matroids. In Reg, the sums $1 + 1$ and $-1 - 1$ are not defined.
2. The partial field NR $= (\{\pm\alpha^i(\alpha - 1)^j : i, j \in \mathbb{Z}\}, \mathbb{Q}(\alpha))$, which leads to the class of near-regular matroids.
3. The partial field $(G_6, \mathbb{C})$, where G_6 is the group of complex sixth roots of unity, which leads to the class of $\sqrt[6]{1}$-matroids.
4. The partial field P_T, defined using the multiplicative group $G_6 = \{a : a^6 = 1\}$. In P_T, addition is only defined when $x + y = 0$, making addition trivial. So we have, $-1 = a^3$, $-a = a^4$ and $-a^2 = a^5$.
5. The partial field P_6, obtained by embedding G_6 as a subgroup of the complex numbers. This extends the partial addition in P_T, allowing operations such as $a^2 + 1 = a$ and $a^4 + 1 = a^5$ as well.
6. A partial field obtained by embedding G_6 into the multiplicative group of $GF(7)$, which results in the field $GF(7)$ itself.
7. The partial field $\mathbb{D} = (\{\pm 2^i : i \in \mathbb{Z}\}, \mathbb{Q})$, which leads to the class of dyadic matroids.

How special are these classes of matroids?

- If F is a field other than $GF(2)$ whose characteristic is not 3, then the class of matroids representable over $GF(3)$ and F, is the class of near-regular matroids, the class of dyadic matroids, the class of $\sqrt[6]{1}$-matroids, or the class of matroids obtained by taking direct sums and 2-sums of dyadic matroids and $\sqrt[6]{1}$-matroids.
- The class of matroids representable over a given partial field is minor-closed and is closed under the taking of duals, direct sums, and 2-sums.

Theorem (Tutte,1965).

A matrix over the real numbers is *totally unimodular* if the determinant of every square submatrix is in the set $\{-1, 0, 1\}$. A matroid is *regular* if it can be represented by a totally unimodular matrix. Tutte proved the following characterization of regular matroids:

Theorem (Tutte, 1965). Let M be a matroid. The following are equivalent:

(i) M is representable over both $GF(2)$ and $GF(3)$;

(ii) M is representable over $GF(2)$ and some field $\mathbb{F}$ that does not have characteristic 2;

(iii) M is representable over $\mathbb{R}$ by a totally unimodular matrix;

(iv) M is representable over every field.

Theorem (Whittle, 1997).

Whittle (1995, 1997) proved very interesting results of a similar nature. Here is one representative example. We say that a matrix over the real numbers is *totally dyadic* if the determinant of every square submatrix is in the set $\{0\} \cup \{\pm 2^k \mid k \in \mathbb{Z}\}$.

Theorem (Whittle, 1997). Let M be a matroid. The following are equivalent:

(i) M is representable over both $GF(3)$ and $GF(5)$;

(ii) M is representable over $\mathbb{R}$ by a totally dyadic matrix;

(iii) M is representable over every field that does not have characteristic 2.

Theorem 3.6

Statement

Let A be a $\mathbb{P}$-matrix whose columns are labelled by a set S. Then the independent subsets of S are the independent sets of a matroid on S.

Proof. Evidently the empty set is independent. Say I is a nonempty independent subset of S with $|I| = k$. By pivoting, taking scalar multiples, interchanging rows and columns, and applying Proposition 3.5, we may assume without loss of generality that the first k rows of the submatrix of columns labelled by I form an identity matrix. All other rows of this submatrix consist of zeros. It follows immediately that all subsets of I are independent.
Now say J is an independent subset of S with $|J| > |I|$. It is easily seen that at least one of the columns labelled by $x \in J$ has a non-zero entry in a row other than the first k rows. Certainly $x \notin I$. It now follows readily that $I \cup \{x\}$ is independent and the theorem is proved.

Further into Partial Fields...

We can further learn about partial field homomorphisms, equivalent representations of two matroids over a partial field, partial fields supporting a finite group and the dowling group geometries associated with such groups, biased graph and its associated matroids, connectivity fucntion, embedding of partial fields, minimum partial fields supporting a group, excluded minor characterizations.

Open Problems

1. Let I be an ideal of the multivariate polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ such that $x_1, \ldots, x_k$ are units of $R := \mathbb{Z}[x_1, \ldots, x_k]/I$, and let G be the subgroup of R^* generated by $x_1, \ldots, x_k$. Is there an algorithm to determine, for $p \in G$, if $1 - p \in G$?
2. Suppose $\mathbb{P}'$ is an induced sub-partial field of $\mathbb{P}$ with $\mathcal{F}(\mathbb{P}') - \{0, 1\} \neq \emptyset$. Are there rings $R, R' \subseteq R$, and groups $G \subseteq R^*, G' \subseteq G$ such that $\mathbb{P}' \cong (R, G)$, $\mathbb{P} \cong (R', G')$, and $G' = G \cap R'$?
3. To what extent is a partial field $\mathbb{P}$ determined by the set of finite fields $GF(q)$ for which there exists a homomorphism $\varphi : \mathbb{P} \to GF(q)$? Remarks: $\mathbb{P}$ is certainly not uniquely determined: both K_2 and U_2 have homomorphisms to all finite fields with at least 4 elements, but some K_2-representable matroids are not U_2-representable. Let $\mathcal{P}_0 := \{0\} \cup \{x \in \mathbb{N} \mid x \text{ is prime}\}$.
4. Let $\mathbb{P}$ be a partial field. The characteristic set of $\mathbb{P}$ is :

$$\chi(\mathbb{P}) := \{p \in \mathcal{P}_0 \mid \exists \text{ a homomorphism } \mathbb{P} \to \mathbb{F} \text{ of characteristic } p\}.$$

For which subsets S of $\mathcal{P}_0$ does there exist a partial field $\mathbb{P}$ with $\chi(\mathbb{P}) = S$?

References

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