The unbreakable quasi-graphic matroids

Sayantani Bhattacharya

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Auburn University

With John David Clifton and Zach Walsh

Outline

Matroids

Definition (Whitney 1935)

A *matroid* M is a pair $(E, \mathcal{C})$. The set E is the *ground set* of M and $\mathcal{C} \subseteq 2^E$ is a collection of *circuits* of M such that

(C1) $\varnothing \notin \mathcal{C}$.

(C2) If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

If a subset $X \subseteq E$ contains an element of $\mathcal{C}$, then X is said to be *dependent*, and is otherwise *independent*.

Graphic Matroids

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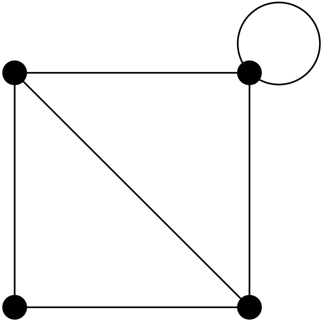

A *loop* of a matroid is a 1-element circuit.

Graph Connectivity vs Matroid Connectivity

A matroid is *connected* if every pair of elements is contained in a common circuit. (Whitney, 1935)

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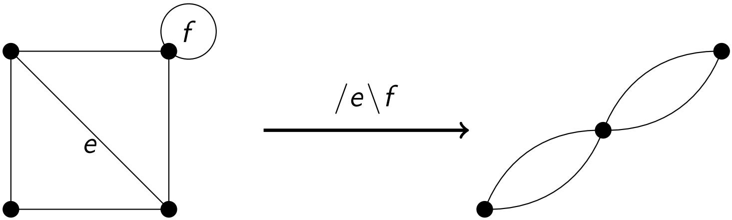
Connectivity of Graphic Matroids (Tutte 1966)

Let G be a loopless graph without isolated vertices and suppose that $|V(G)| \geq 3$. Then $M(G)$ is a connected matroid if and only if G is a 2-connected graph.

Matroid Minors

Matroids are also equipped with *deletion* and *contraction* operations which generalize graph minor operations.

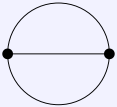


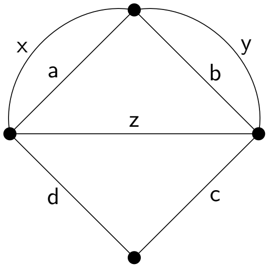
Deletion and contraction operations commute both with each other and with themselves.

Biased Graphs

Definition (Zaslavsky 1989)

A *biased graph* is a pair $(G, \mathcal{B})$ consisting of a graph G and a collection $\mathcal{B}$ of 'balanced cycles' of G satisfying the theta property: no theta subgraph (or subdivision) of G contains exactly two cycles in $\mathcal{B}$.





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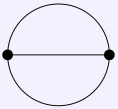

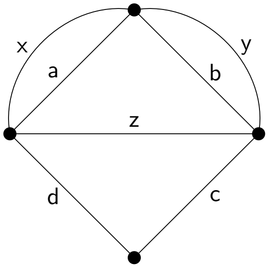

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Partition of Cycles of Biased-graphic Matroids

Zaslavsky's converse question : Given a group-labelled graph G when can we define a matroid $M(G, \mathcal{B})$ on it?

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Balanced ($\mathcal{B}$) vs Unbalanced ($\mathcal{U}$) Cycles

Let $(\mathcal{B}, \mathcal{U})$ be the partition of cycles of a group-labelled graph G into two sets. Call the cycles whose edge set in $M(G, \mathcal{B})$ is a circuit (minimal dependent set) to be a 'balanced cycle' ($\mathcal{B}$) of G. Otherwise, if the edges form an independent set, it is an 'unbalanced cycle' ($\mathcal{U}$) of G.

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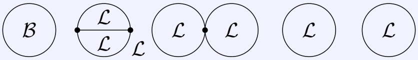
Sufficient Condition : $\mathcal{B}$ satisfies theta property.

Lift Matroids

Definition (Zaslavsky 1991)

A *lift matroid* $M = L(G, \mathcal{B})$ is a matroid associated with a biased graph $(G, \mathcal{B})$, whose circuits are subdivisions of the following subgraphs:



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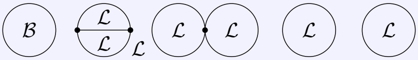
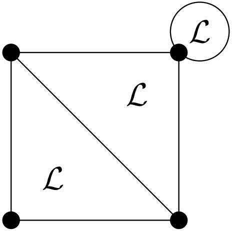


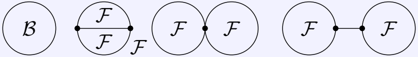
When all cycles are balanced, M is a graphic matroid.

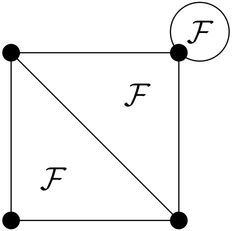


Frame Matroids

Definition (Zaslavsky 1994)

A *frame matroid* $M = F(G, \mathcal{B})$ is a matroid associated with a biased graph $(G, \mathcal{B})$, whose circuits are subdivisions of the following subgraphs:





Quasi-Biased Graphs

Definition (Bowler, Funk and Slilaty 2020)

Given a biased graph $(G, \mathcal{B})$, we 'refine' the collection of cycles $(\mathcal{B}, \mathcal{U})$ to a *proper tripartition* $(\mathcal{B}, \mathcal{L}, \mathcal{F})$ so that $\mathcal{B}$ satisfies theta property and every cycle in $\mathcal{L}$ meets every cycle in $\mathcal{F}$ at a common vertex.

A graph G together with such a proper tripartition $(G, \mathcal{B}, \mathcal{L}, \mathcal{F})$ of its cycles, is called a *quasi-biased graph*.

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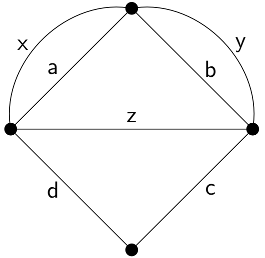
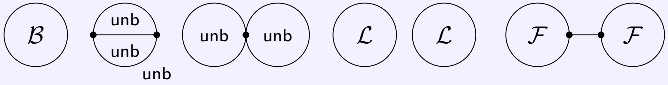


$\mathcal{B} = \{\{abz\}, \{cdz\}, \{xyz\}, \{abcd\}, \{cdxy\}\}$
$\mathcal{L} = \{\{ax\}, \{ayz\}\}$
$\mathcal{F} = \{\{by\}, \{bxz\}, \{aycd\}, \{bcdx\}\}$

Quasi-graphic Matroids

Definition

Every biased graph with a proper tripartition has an associated *quasi-graphic* matroid $M = M(G, \mathcal{B}, \mathcal{L}, \mathcal{F})$, whose circuits are subdivisions of the following subgraphs:



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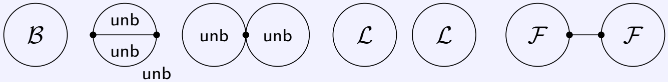

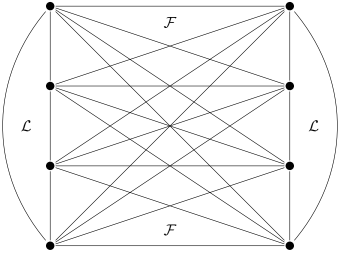

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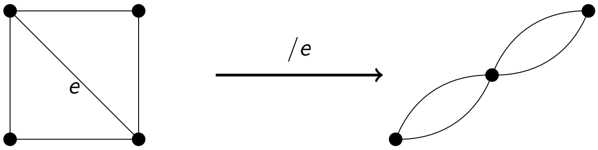

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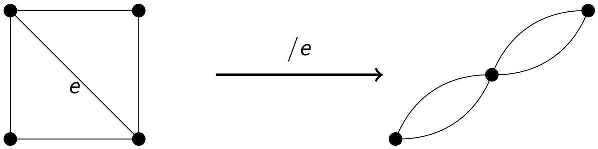

Theorem (Oxley, Pfeil 2016)

A graphic matroid M on graph G is unbreakable if and only if G is loopless and $\mathrm{si}(G)$ is a complete graph or a cycle on at least three vertices.

3-connected Unbreakable Matroids

Let G be a graph without isolated vertices and suppose that $|E(G)| \geq 4$. Then $M(G)$ is *3-connected* if and only if G is 3-connected and simple.

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Theorem (Fife, Mayhew, Oxley, Semple 2020)

Let $F(G, \mathcal{B})$ be a 3-connected unbreakable frame matroid where G has no isolated vertices. Then either $|V(G)| \leq 6$ or $\mathrm{si}(G)$ is obtained from a complete graph by deleting the edges of a path of length at most 2.

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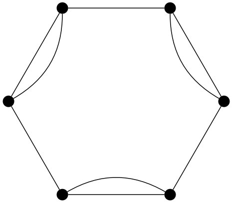

3-connected Unbreakable Matroids

Theorem (B., Clifton, Walsh 2026+)

Let $M(G, \mathcal{B}, \mathcal{L}, \mathcal{F})$ be a 3-connected unbreakable quasi-graphic matroid where each component of G has at least two vertices. Then either $|V(G)| \leq 6$, or $\text{si}(G)$ is a cycle, or $\text{si}(G)$ is obtained from a complete graph by deleting a path of length at most 2.

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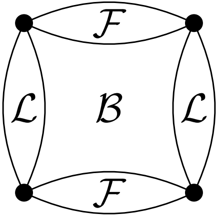

Proof Sketch

Our proof generalizes Fife et al.'s previous result.

Two cases:

1. G is 2-connected but not 3-connected.

   $\implies$ either $|V(G)| \leq 6$ or $\mathrm{si}(G)$ is a cycle and M is lifted-graphic.

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2. G is 3-connected and $|V(G)| \geq 7$.

   $\implies$ $\mathrm{si}(G)$ can be obtained from a complete graph by deleting the edges of a path of length at least two.

Future Direction

1. Extend these results to biased graphic matroids which are not quasi-graphic.
2. Extend these results for binary and representable matroids.

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https://arxiv.org/abs/2605.12811

THANK YOU